# Boosting STAT 4710

November 7, 2023

#### Where we are

Unit 1: R for data mining
Unit 2: Prediction fundamentals
Unit 3: Regression-based methods
Unit 4: Tree-based methods
Unit 5: Deep learning

Lecture 1: Growing decision trees

Lecture 2: Tree pruning and bagging

Lecture 3: Random forests

Lecture 4: Boosting

Lecture 5: Unit review and quiz in class



Looking back:

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- Boosting: Grow shallow decision trees sequentially

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Final prediction rule:  $\hat{f} = \hat{f}_1 + \cdots + \hat{f}_R$ .

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Final prediction rule:  $\hat{f} = \lambda \hat{f}_1 + \cdots + \hat{\lambda}$ 

$$\lambda \hat{f}_B$$



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Large enough B can lead to overfitting, unlike random forests.



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$$\hat{f}(X) = \hat{g}_1(X_1) + \hat{g}_2(X_2) + \cdots + \hat{g}_n(X_n) +$$

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The coordinate functions can be easily plotted and interpreted.







 $C_2$ 

#### **Derivation of coordinate functions**

Get coordinate functions by grouping stumps splitting on the same variable:

 $\hat{f}(X_1, X_2) = \hat{f}_1(X_1) + \hat{f}_2(X_2) + \hat{f}_3(X_1) + \hat{f}_4(X_1) + \hat{f}_5(X_2)$
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- Subsampling empirically demonstrated to improve boosting performance.
- Subsampling increases variance of individual trees but de-correlates them; benefit of the latter tends to outweigh the former.
- A subsampling fraction of  $\pi = 0.5$  tends to work well in most cases.



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- The key issue is that there is not an obvious notion of residual in classification.
- Implementation of boosting for classification is beyond the scope of the class, but the same intuitions from this lecture carry over to boosting for classification.



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• Purity-based variable importance as well as partial dependence plots help