Lasso regression

STAT 4710

October 17, 2023

Where we are

Unit 1: R for data mining

Unit 2: Prediction fundamentals

Unit 3: Regression-based methods

Unit 4: Tree-based methods

Unit 5: Deep learning

Lecture 1: Linear and logistic regression

Lecture 2: Regression in high dimensions

Lecture 3: Ridge regression

Lecture 4: Lasso regression

Lecture 5: Unit review and quiz in class

Idea: Encourage coefficients to be exactly zero

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It turns out that changing the penalty in this way leads to $\hat{\beta}_j^{\text{lasso}} = 0$ for many j.

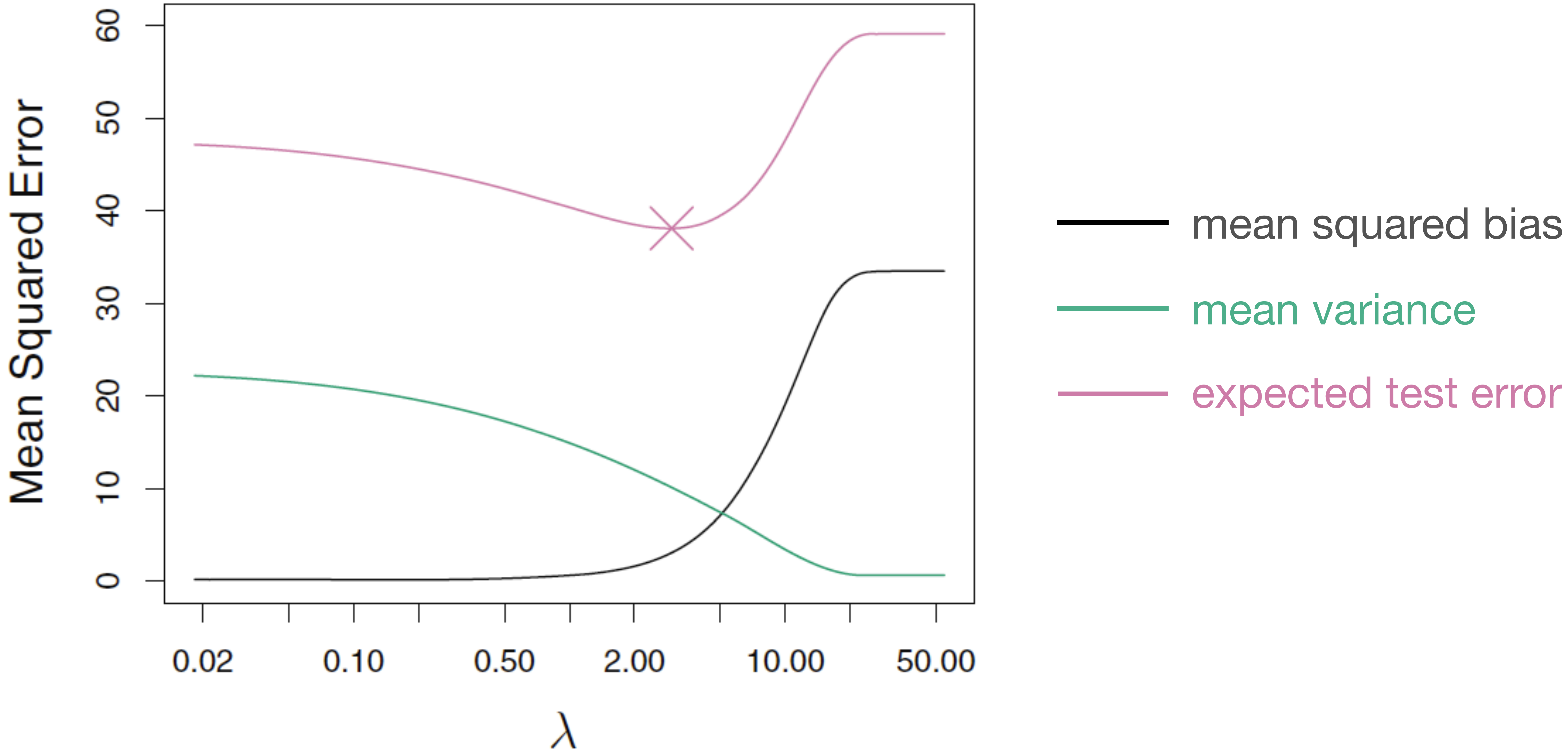
The effect of the penalty parameter λ

$$\hat{\beta}^{\text{lasso}} = \arg\min_{\beta} \left\{ \sum_{i=1}^{n} (Y_i - (\beta_0 + \beta_1 X_{i1} + \cdots + \beta_{p-1} X_{i,p-1}))^2 + \lambda \sum_{j=1}^{p-1} |\beta_j| \right\}.$$

- The larger λ is, the more of a penalty there is.
- For $\lambda = 0$, we get back ordinary least squares (if OLS solution exists)
- For $\lambda = \infty$, we get $\beta_1 = \cdots = \beta_{p-1} = 0$, leaving only the intercept (which is not penalized).

We should think of λ as controlling the flexibility of the lasso regression fit, like the degrees of freedom in a spline fit. However, larger λ means fewer degrees of freedom.

The bias-variance tradeoff for lasso regression



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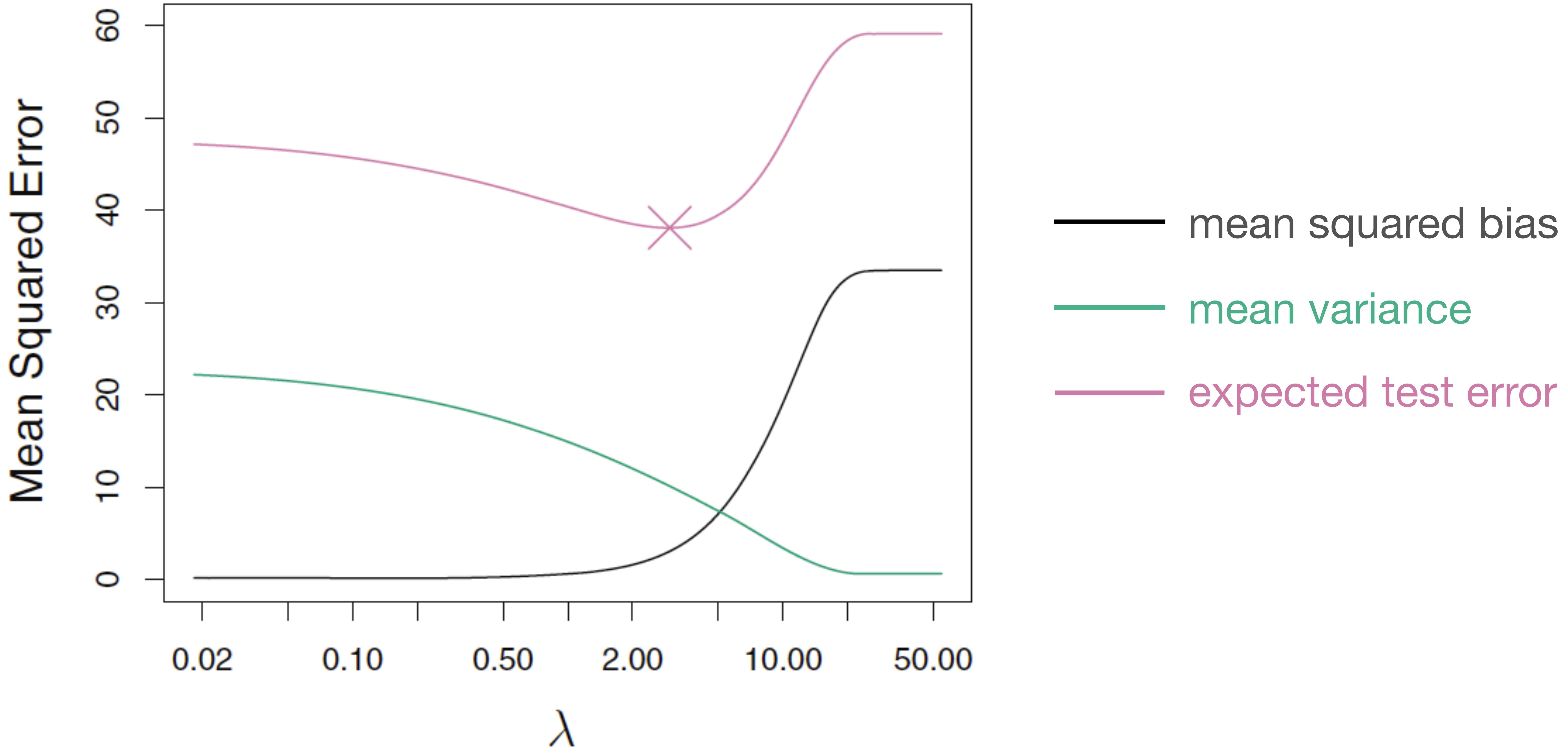

In practice, λ is chosen by cross-validation.

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   <chr>                         <dbl>
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 4 pct.people.dense.hh         10.0
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 6 pct.youngkids2parents       -0.821
 7 num.kids.nvrmarried          0.00737
 8 population                   0
 9 household.size               0
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NOTE: Cannot attach a measure of statistical significance to the selected variables.

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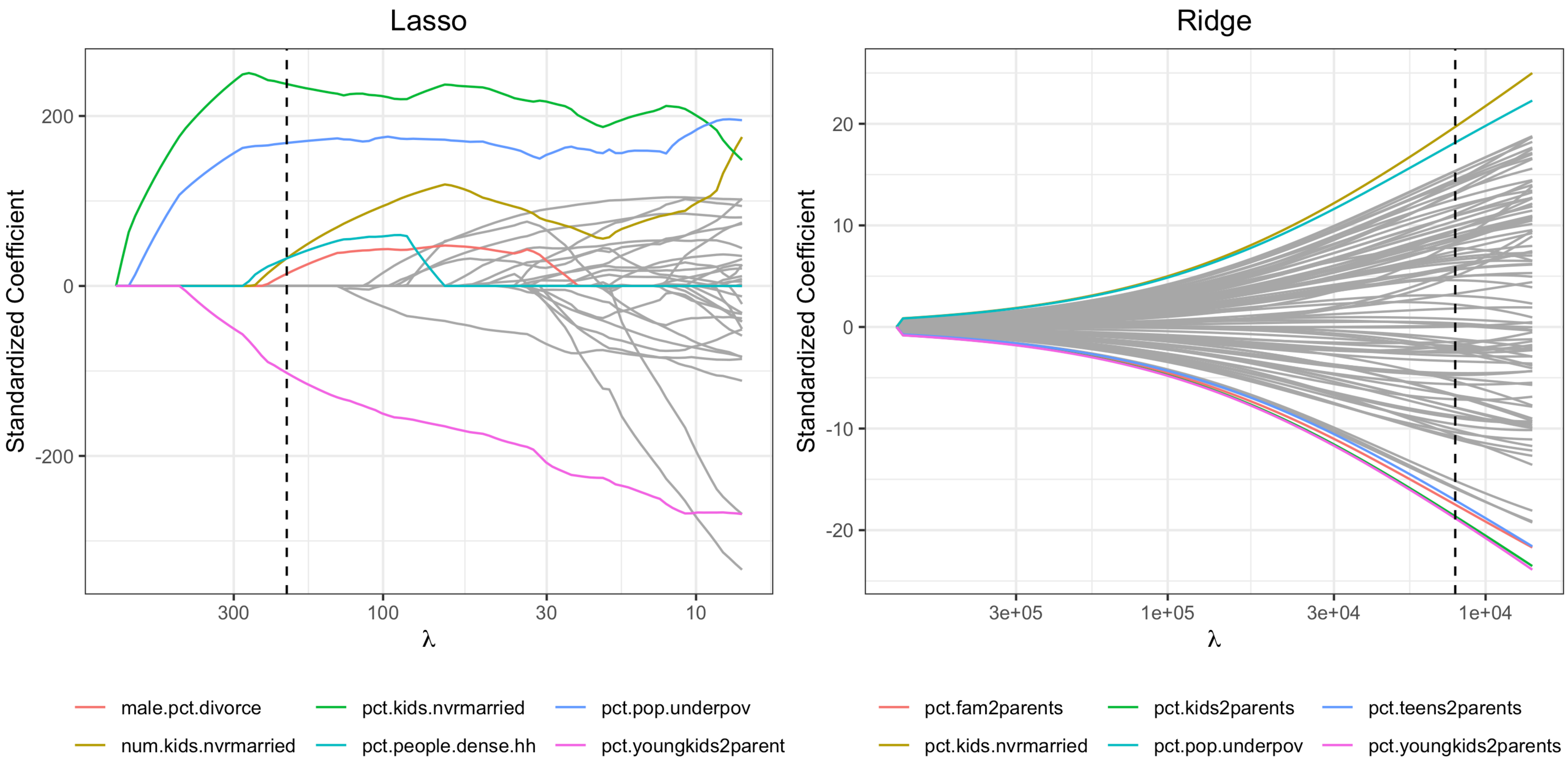
Lasso trace plot (compared to ridge)



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Suppose that $n = p$ and $X_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$, i.e. $Y_j = \beta_j + \epsilon_j$.

E.g. $X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

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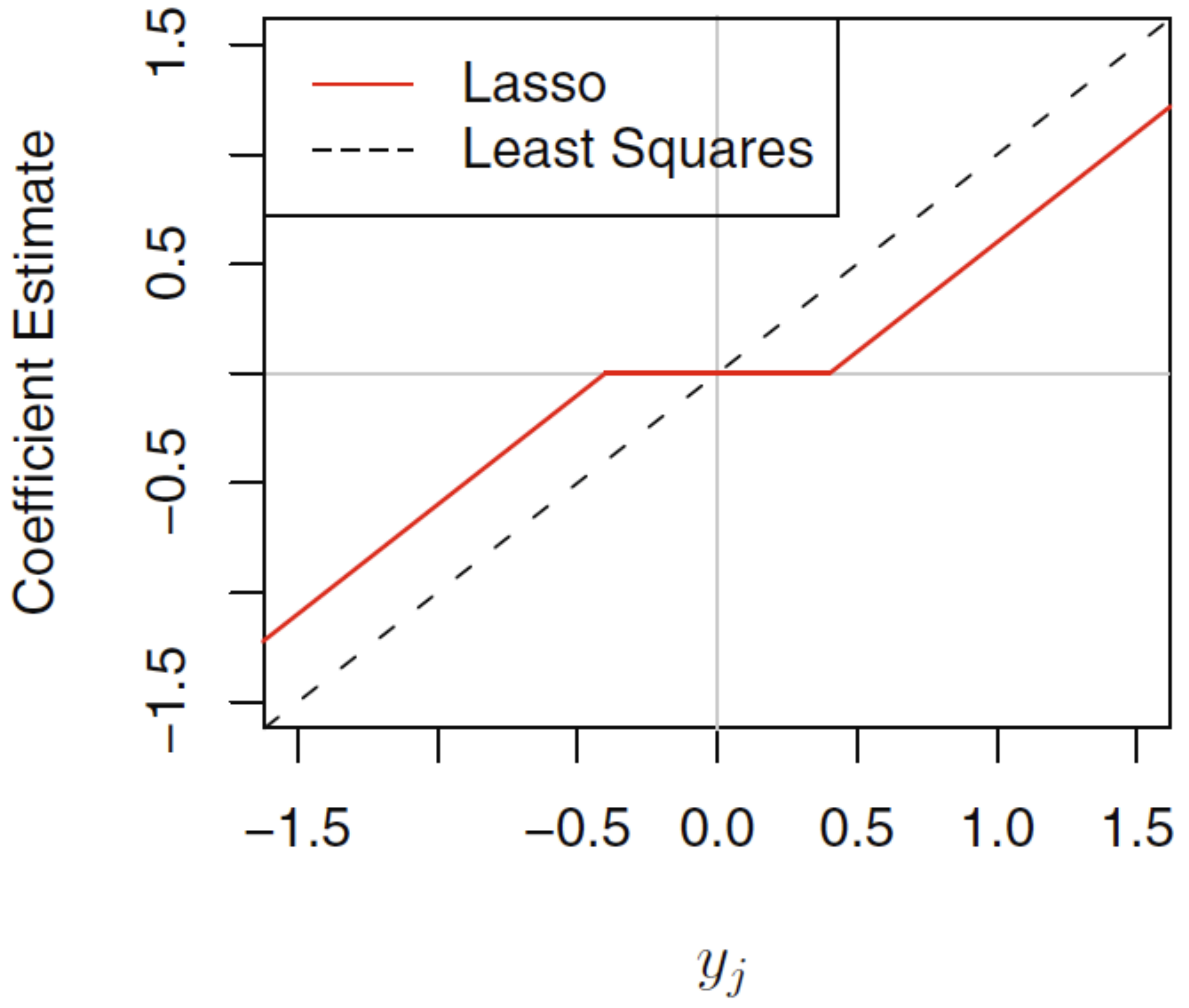

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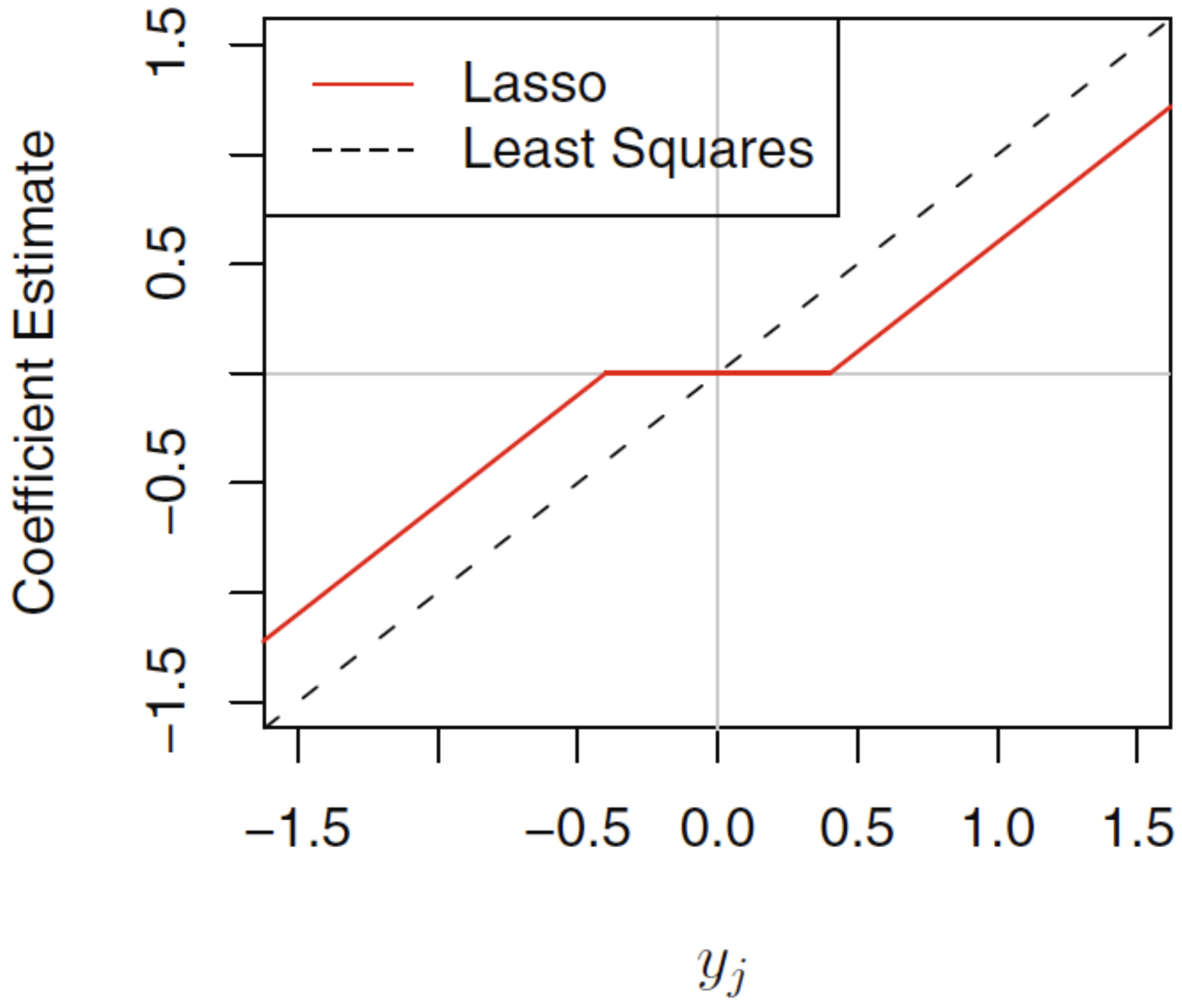

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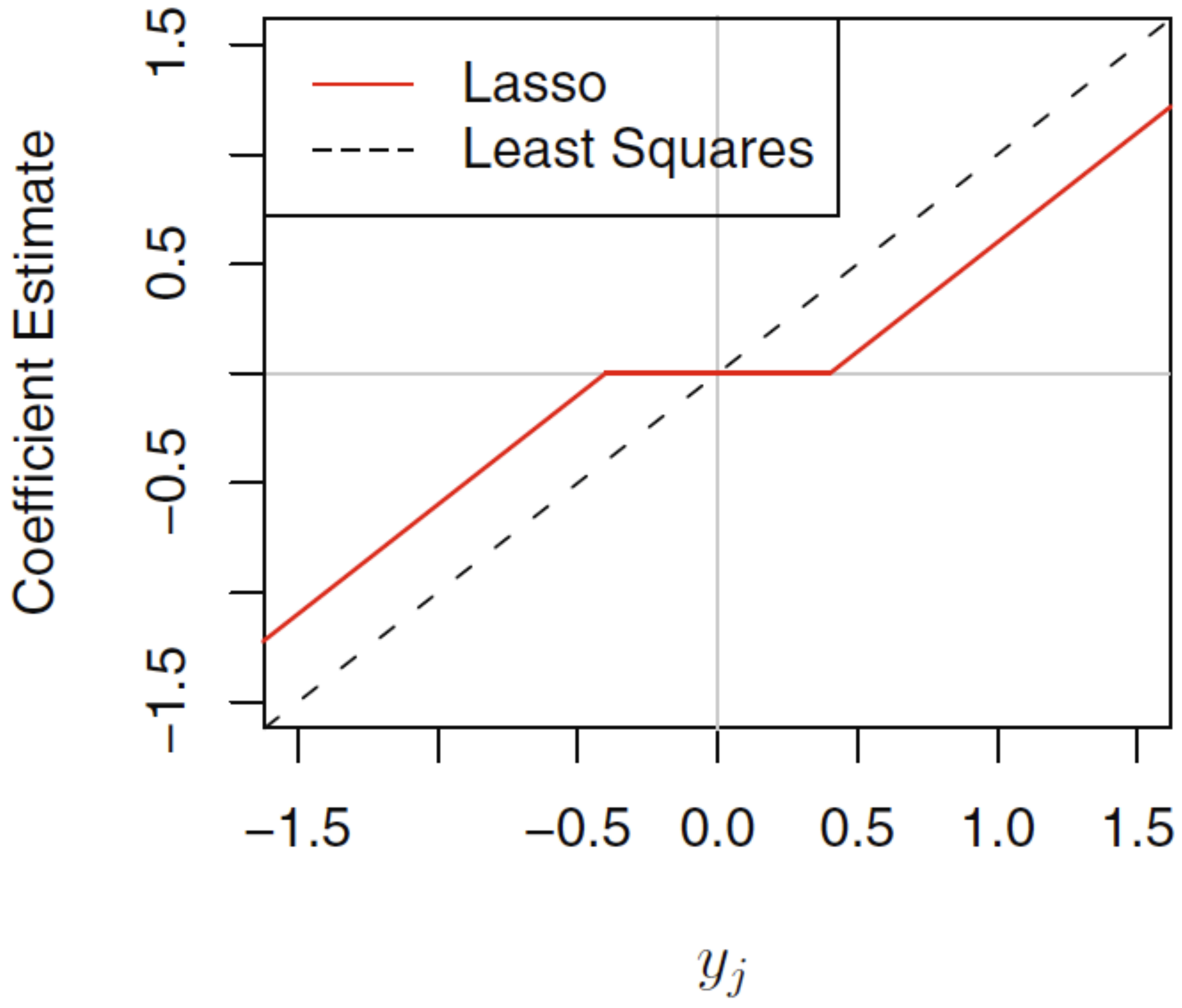

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LASSO = Least Angle Shrinkage and Selection Operator.

Feature scaling and standardization

Like for ridge regression, feature scaling matters for the lasso;

Feature standardization is recommended before running the lasso.

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Note: Coefficient instability doesn't necessarily translate into prediction instability.

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Subtle point: While $\widehat{\beta}^{\text{lasso}}$ is trained based on a (penalized) log-likelihood, during cross-validation we should choose λ based on whatever measure of test error we care about (e.g. weighted misclassification error).

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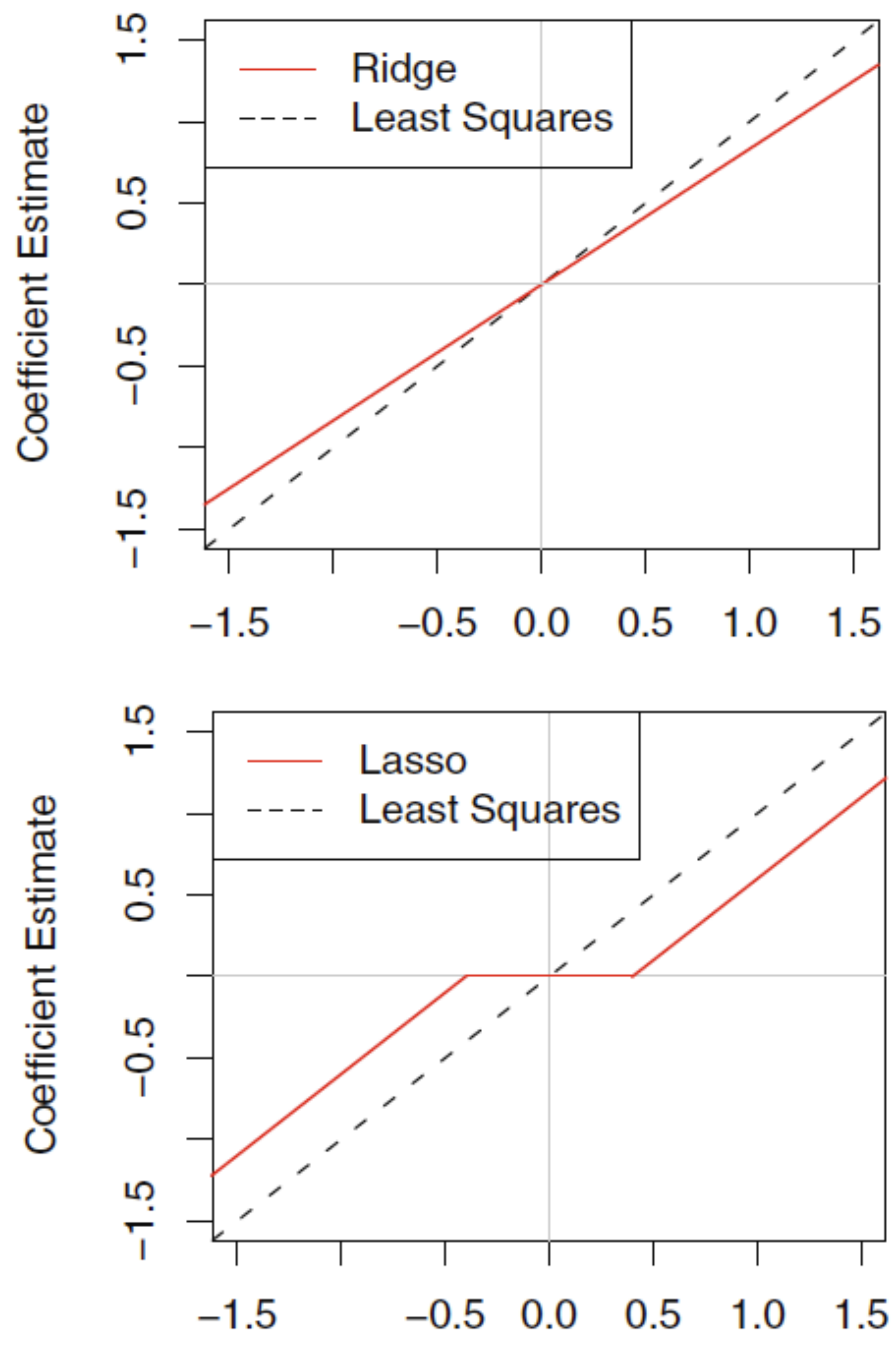

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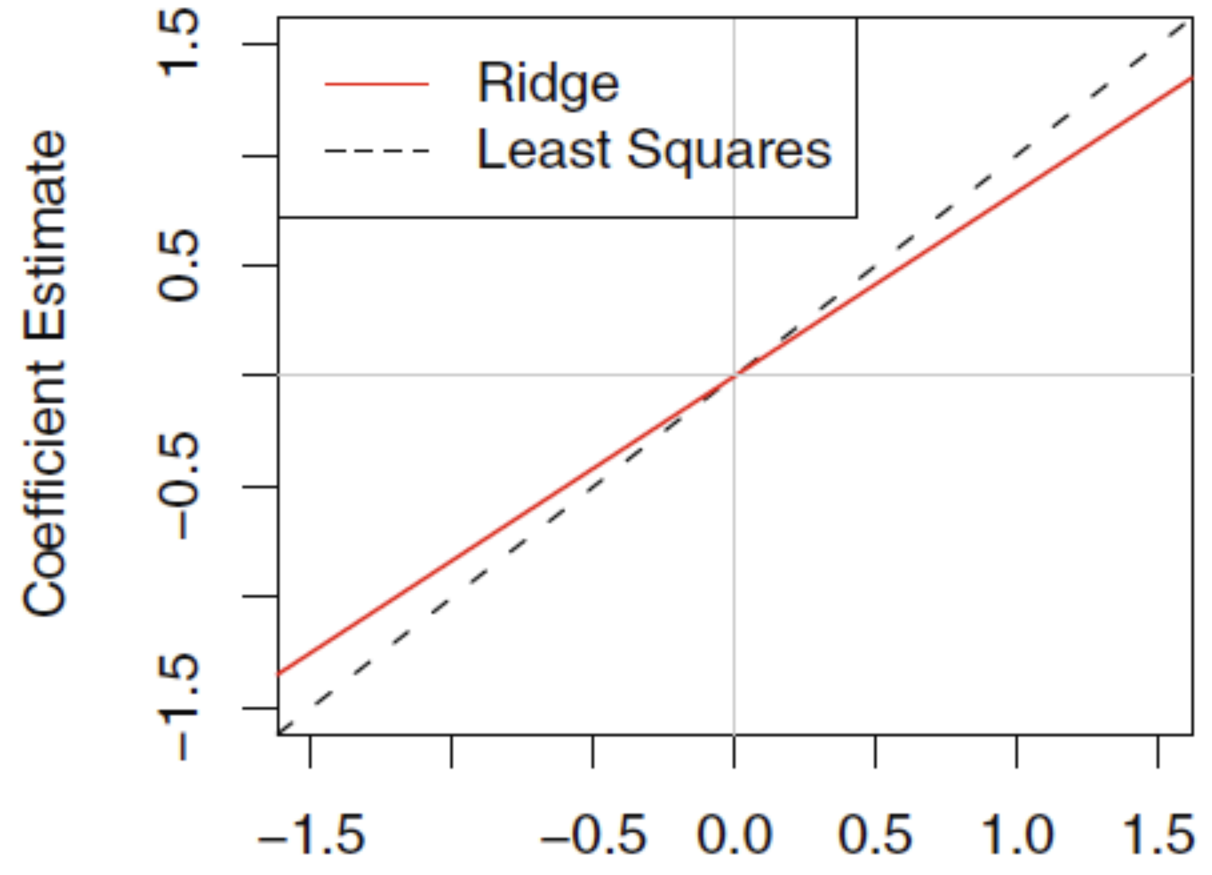

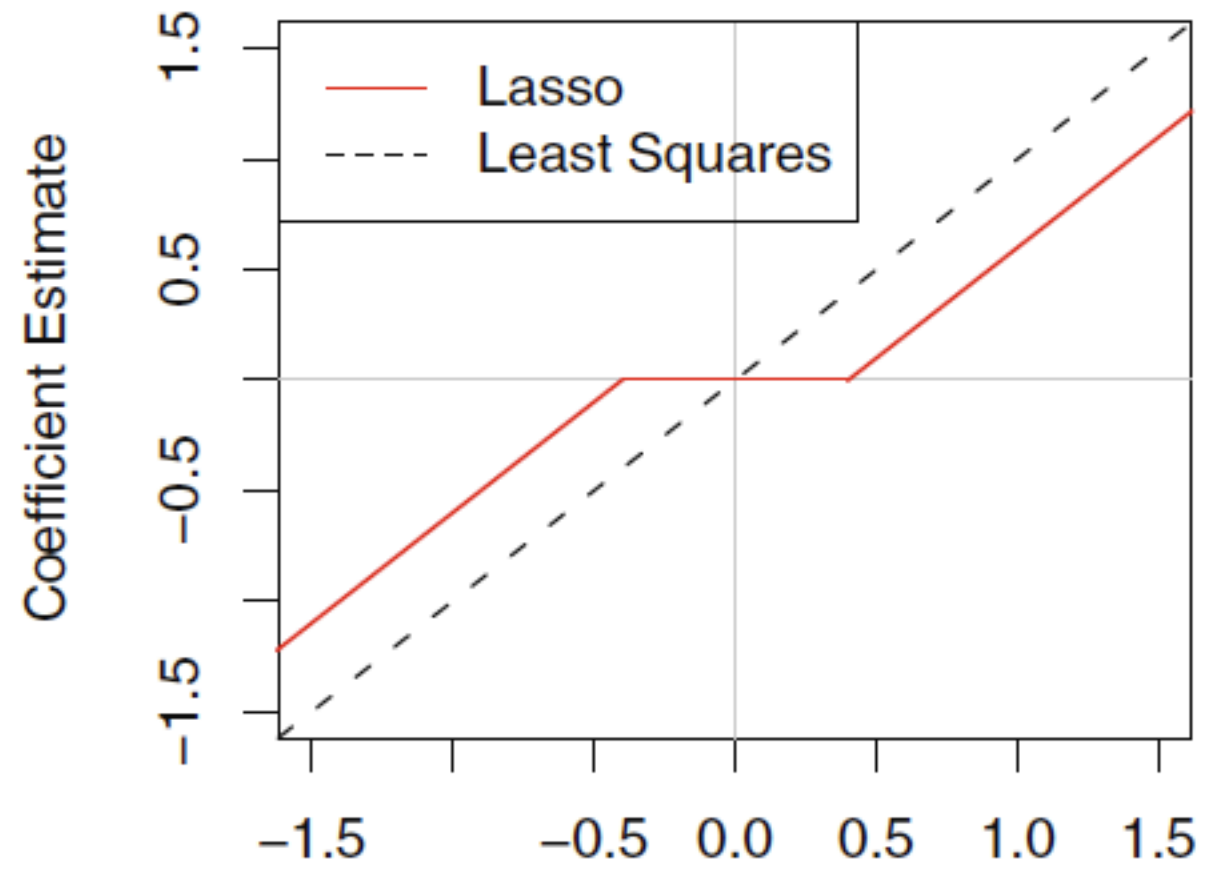

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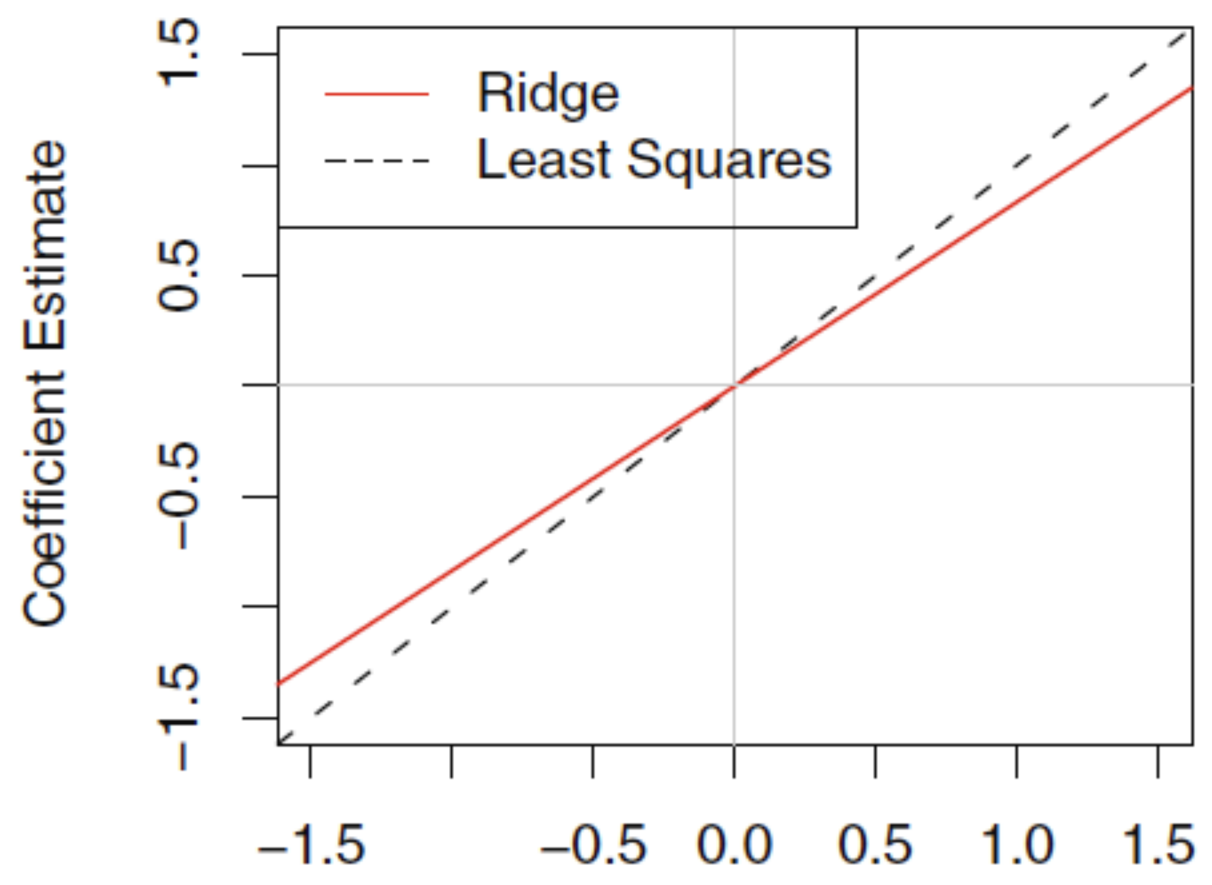

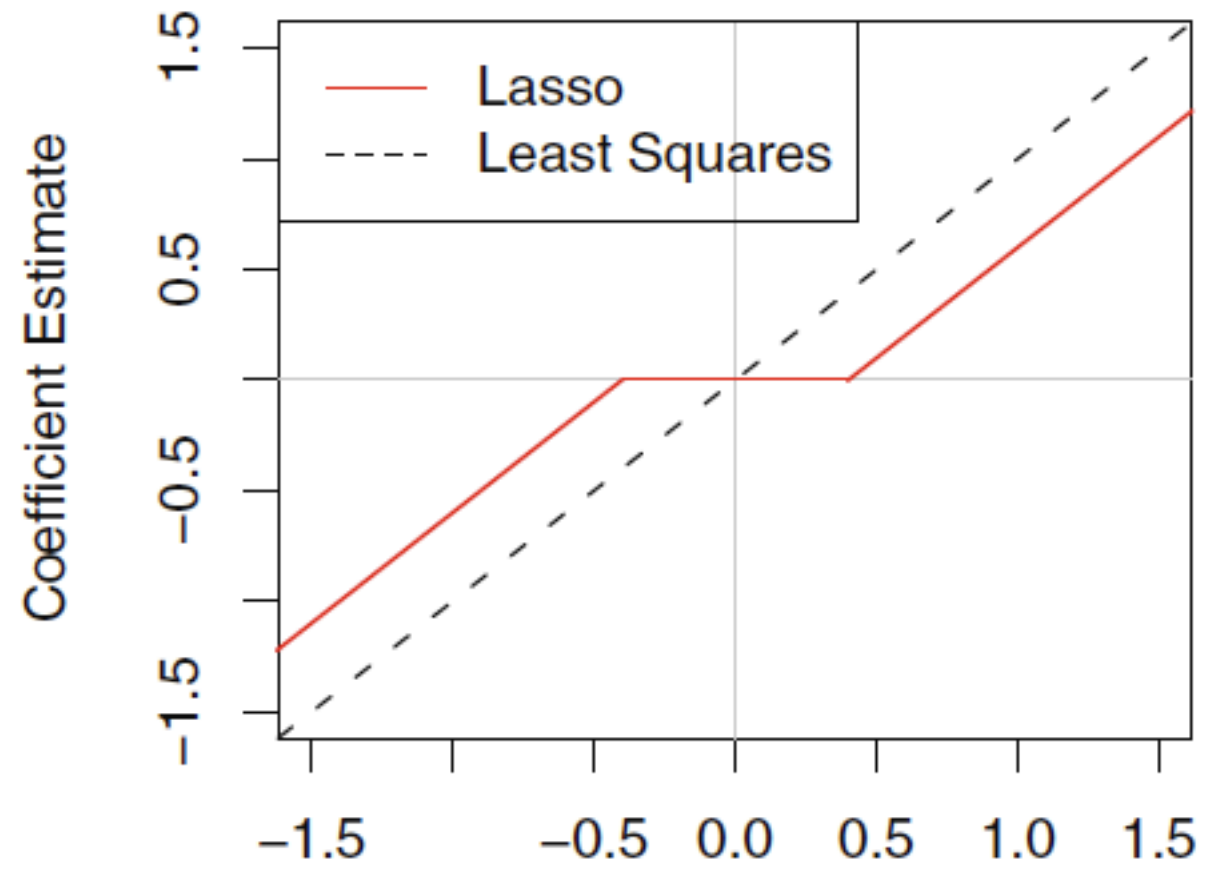

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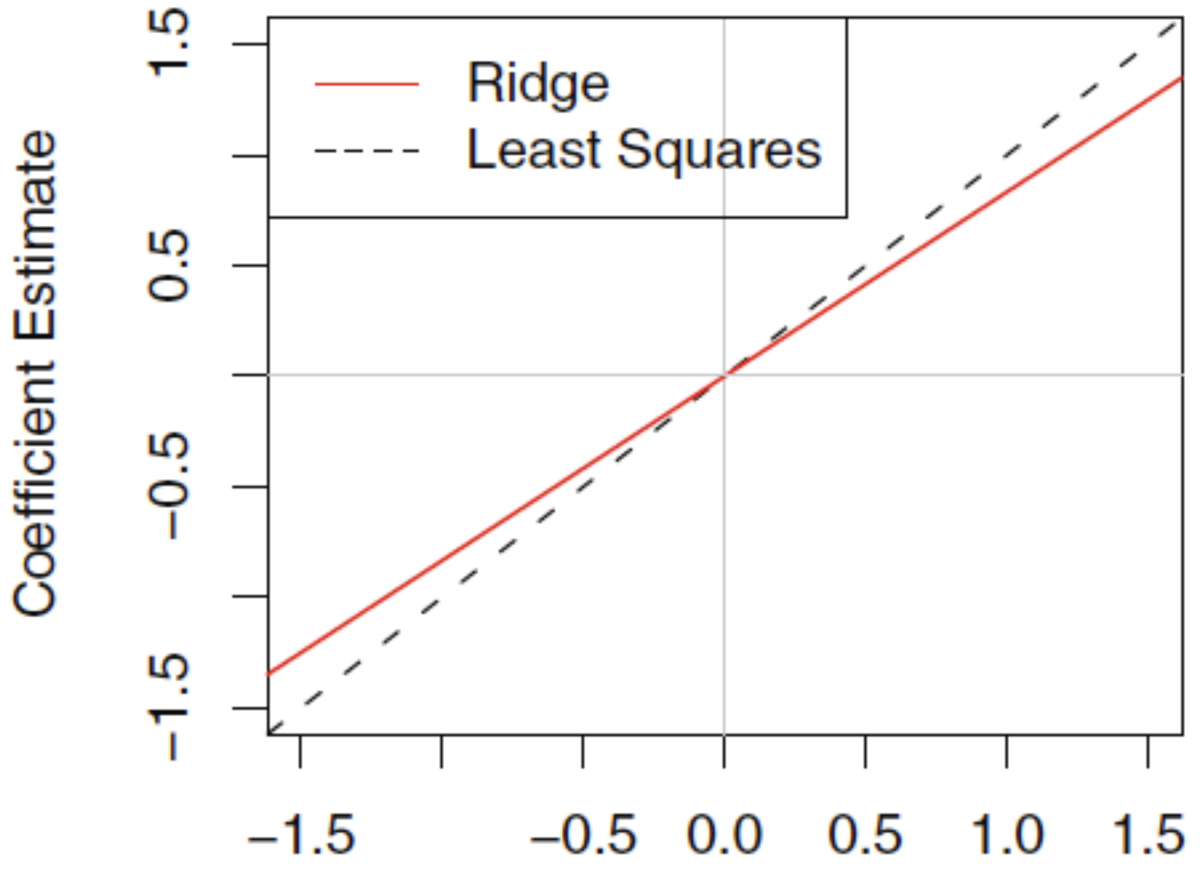

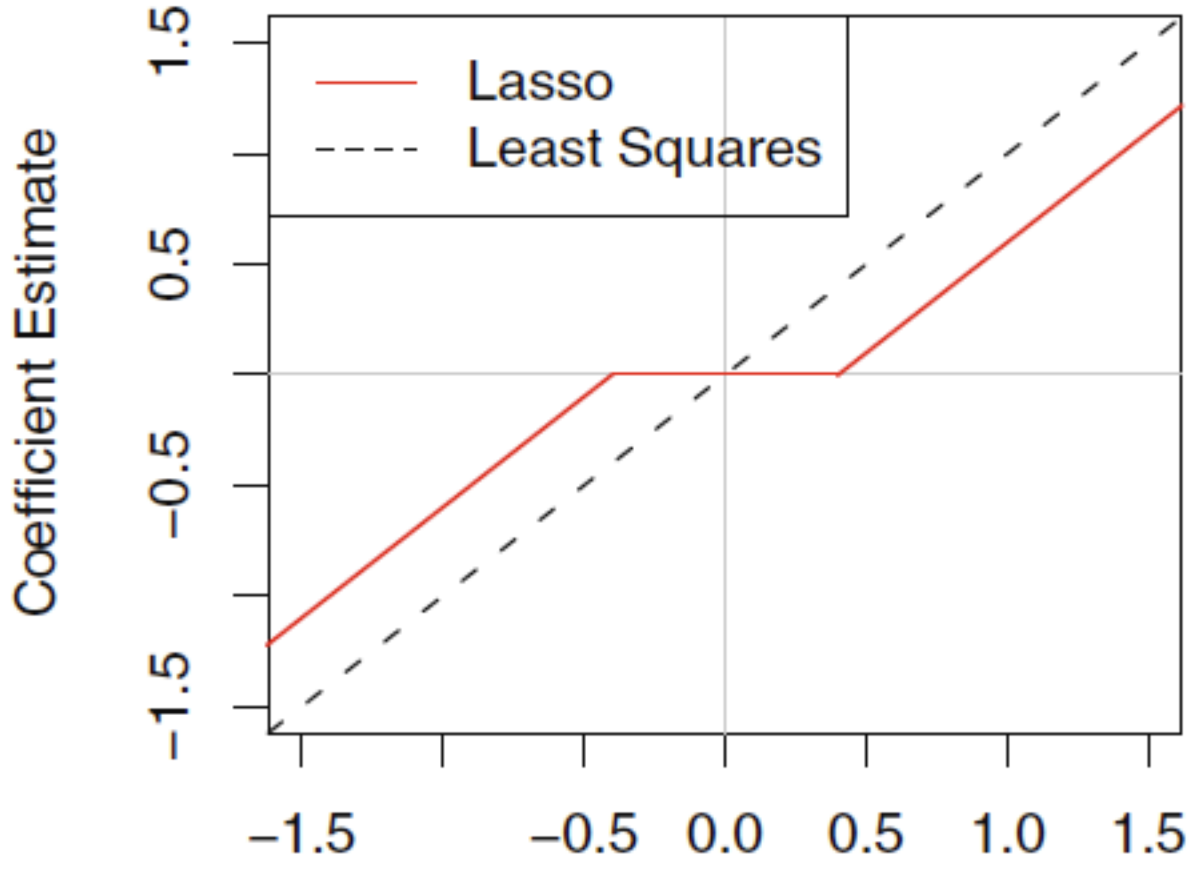

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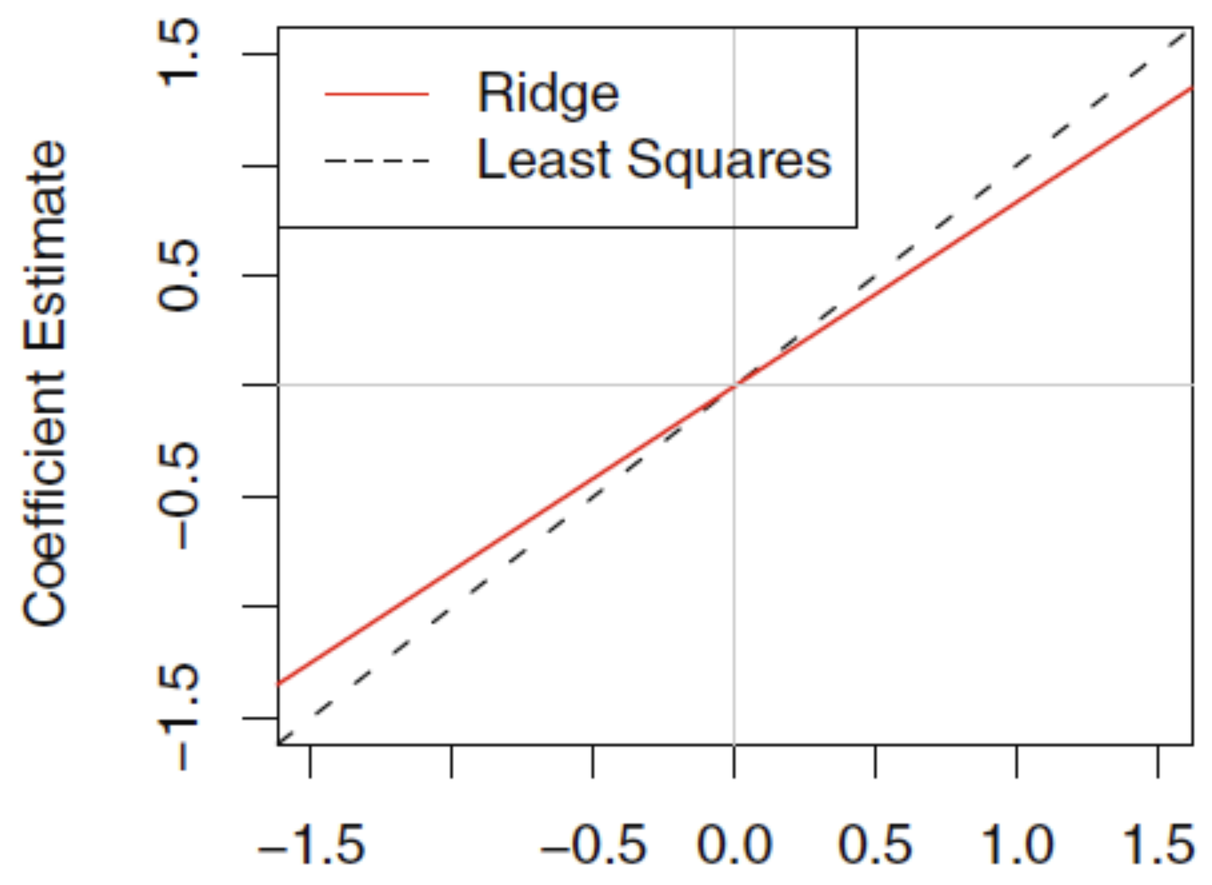
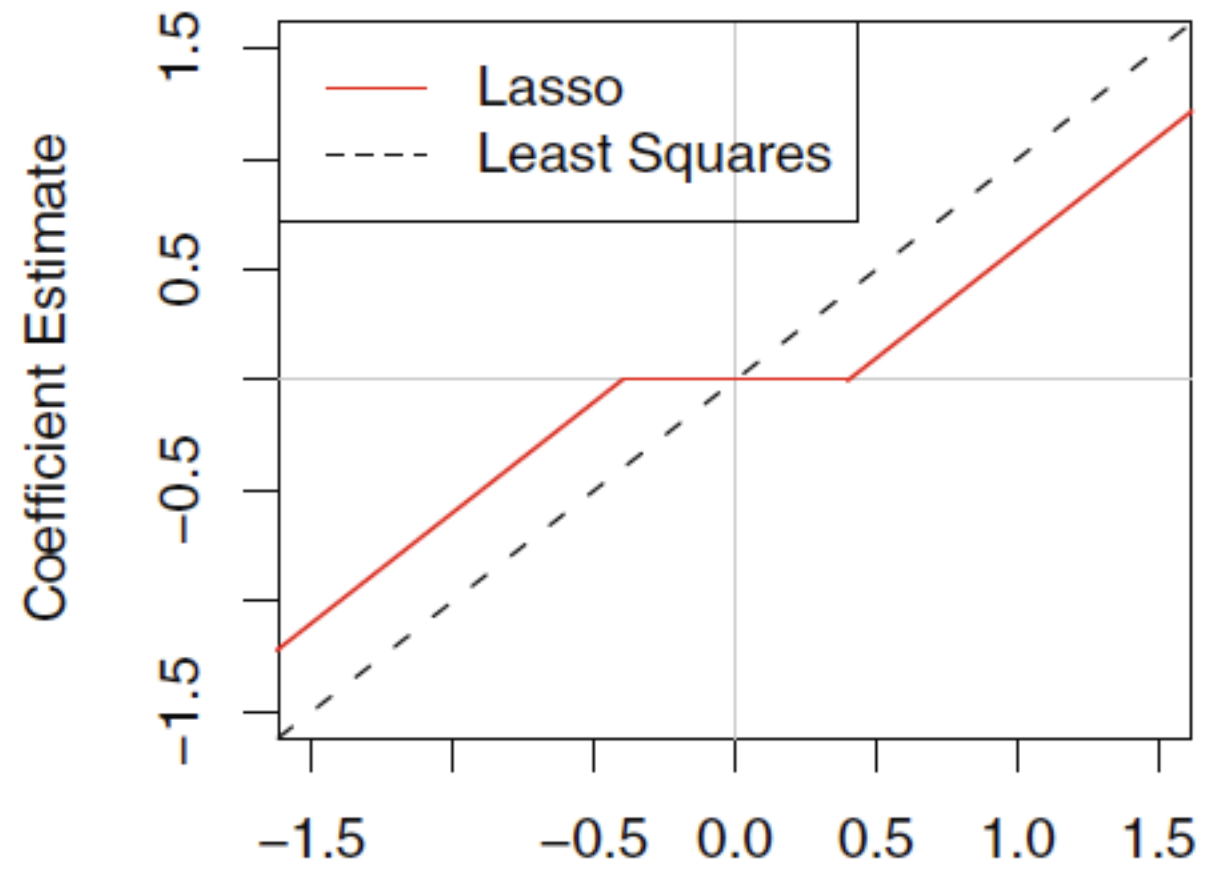
Works when $p > n$	No	Yes	Yes





Elastic net regression

Get the benefits of ridge and lasso regression by combining the two penalties:

$$\text{Penalty} = (1-\alpha)\sum_{j=1}^{p}\beta_j^2 + \alpha\sum_{j=1}^{p}|\beta_j|$$

- When $\alpha = 0$, we get ridge regression
- When $\alpha = 1$, we get lasso regression
- When $0 < \alpha < 1$, we get ridge-like shrinkage as well as lasso-like selection

Elastic net gives sparse solutions as long as $\alpha > 0$.

How to choose α? Can cross-validate over α and λ: First choose α to minimize CV error, then choose λ according to the one-standard-error rule.

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  - Coefficient estimates unstable in the presence of correlated features.

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- Penalized regression method encouraging coefficients to be sparse.
- Like ridge:
  - λ controls the degrees of freedom; *larger* λ gives *fewer* degrees of freedom.
  - Bias-variance trade-off: larger λ gives higher bias but lower variance.
  - Features need to be standardized.
  - Can be applied to logistic regression as well.
- Unlike ridge:
  - Sets some coefficients exactly to zero, achieving variable selection.
  - Coefficient estimates unstable in the presence of correlated features.
- Can be combined with ridge (elastic net).