Regression in high dimensions

STAT 4710

October 5, 2023

Where we are

Unit 1: R for data mining

Unit 2: Prediction fundamentals

Unit 3: Regression-based methods

Unit 4: Tree-based methods

Unit 5: Deep learning

Lecture 1: Linear and logistic regression

Lecture 2: Regression in high dimensions

Lecture 3: Ridge regression

Lecture 4: Lasso regression

Lecture 5: Unit review and quiz in class

High-dimensional data

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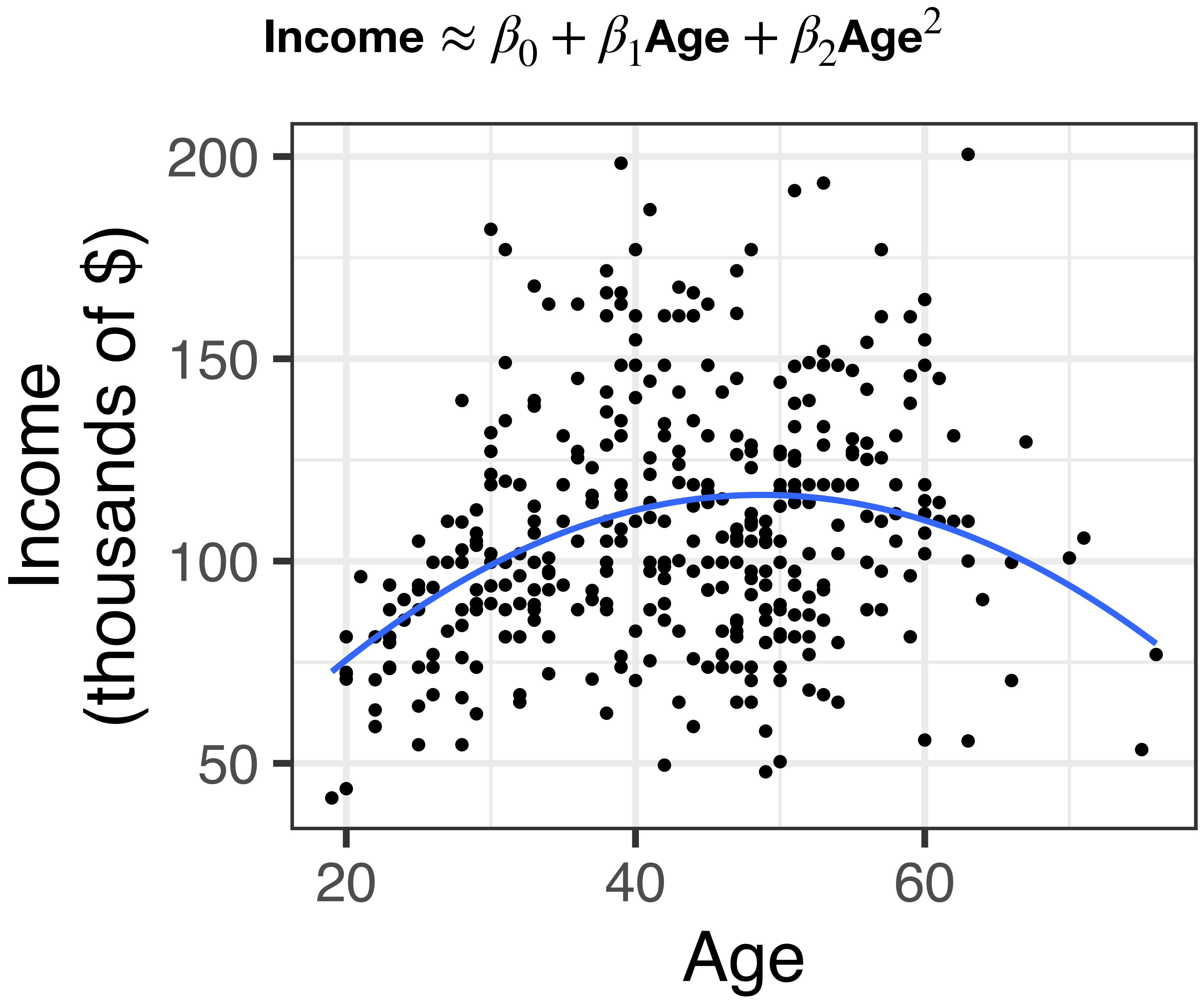

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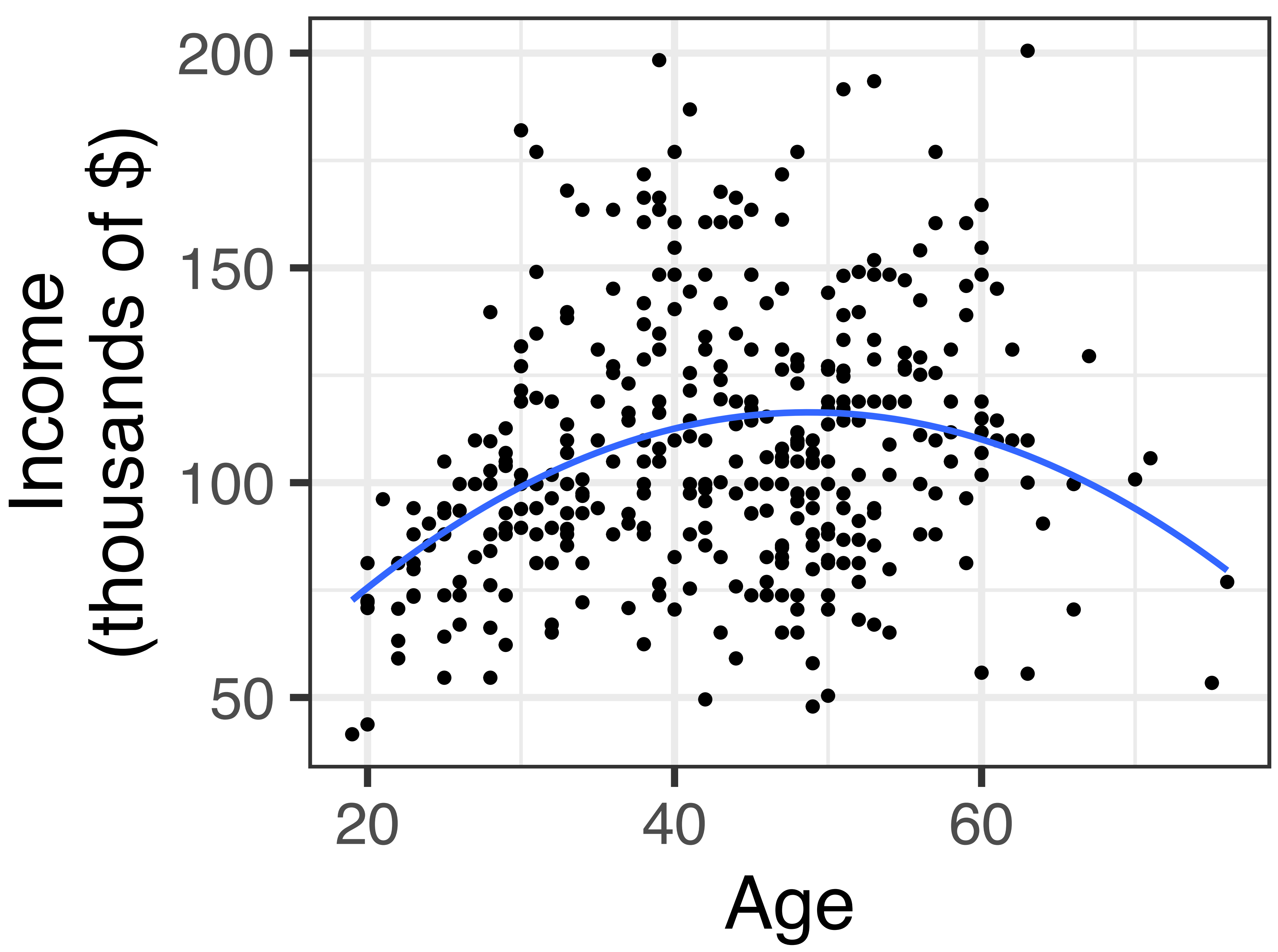
In modern applications, can collect very many features for each observation, e.g.:

- Natural language processing
- Image processing
- Genetics/Genomics
- E-commerce

Income $\approx \beta_0 + \beta_1$**Age** $+ \beta_2$**Age**^{2}



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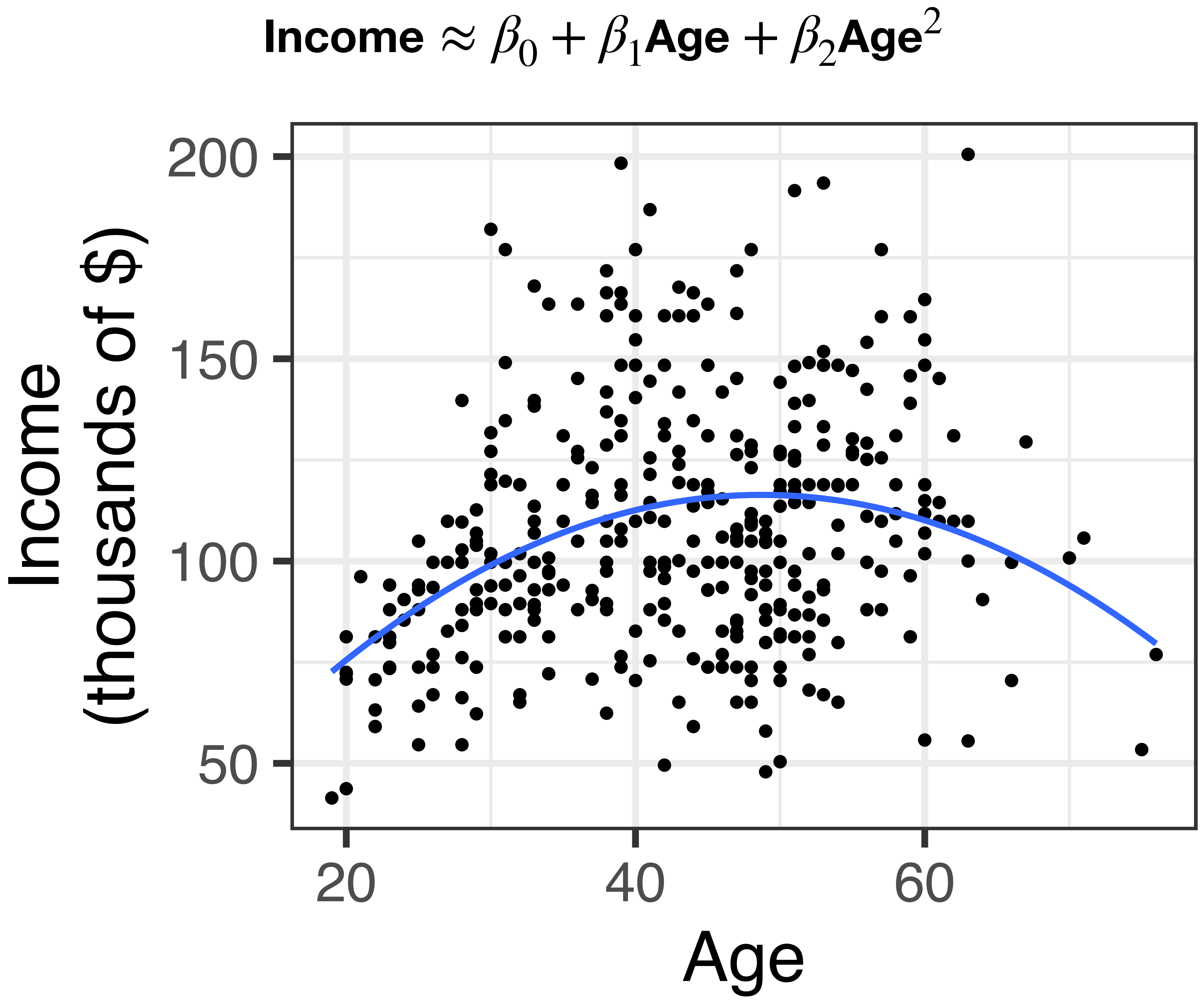

High-dimensional data: Data with $p > n$ or $p \approx n$

Challenges in high dimensions

Let’s consider fitting a linear regression with n observations and p features.

If $p > n$, the columns of the feature matrix X guaranteed to be multi-collinear, so the least squares linear regression estimate is not even defined.

If $p = n$, linear regression will perfectly fit training set, even with “junk” features.

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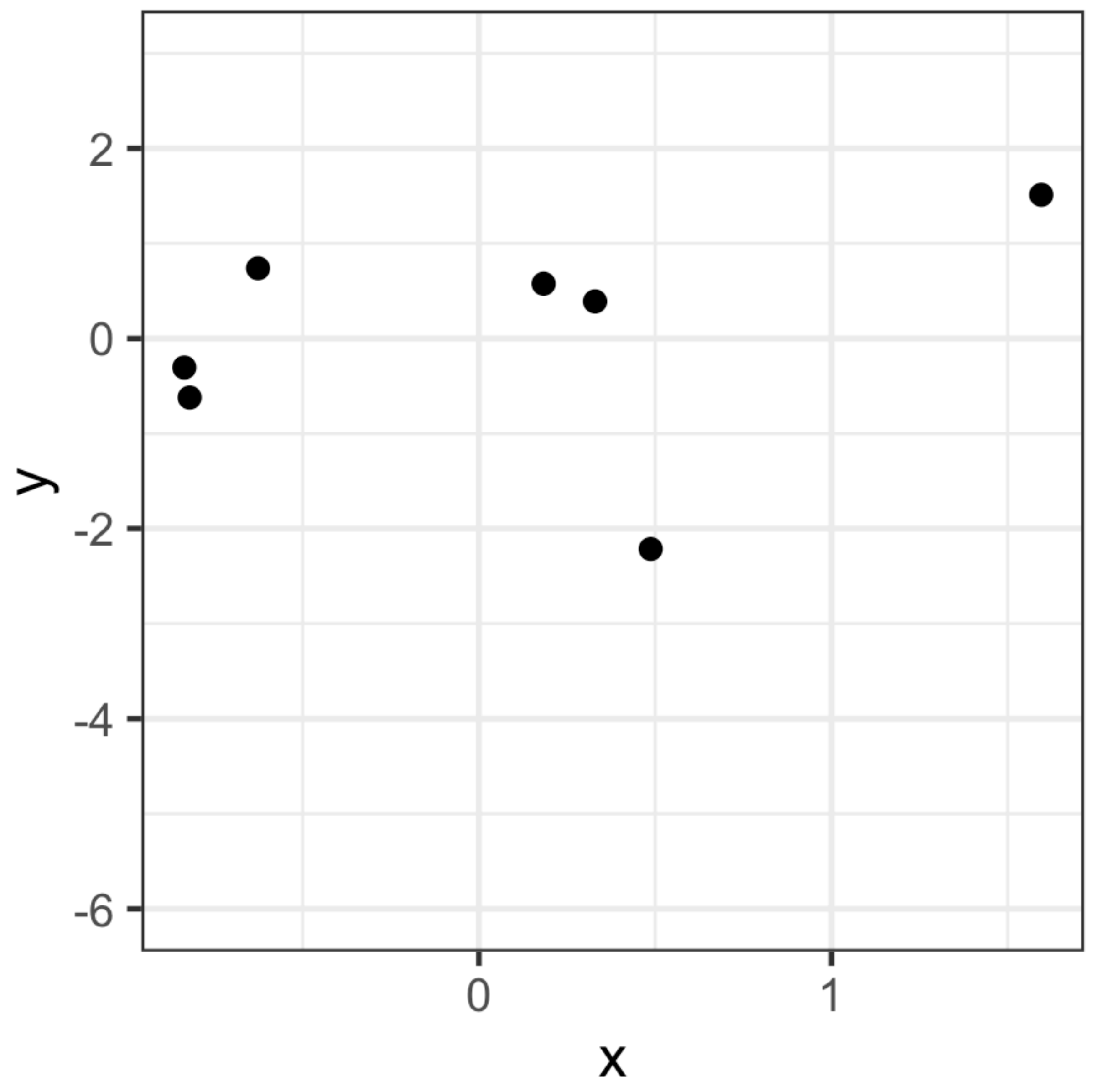

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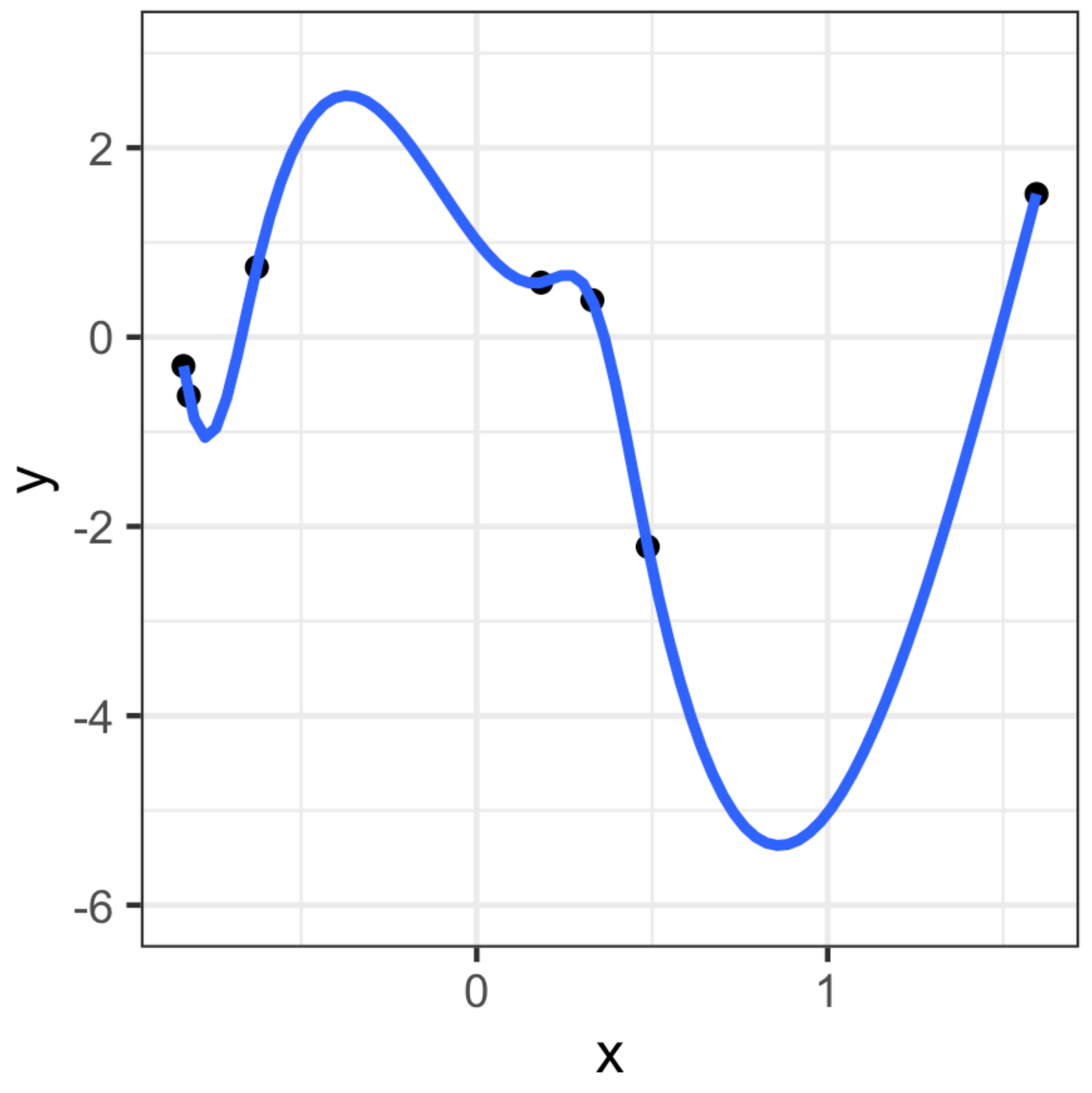

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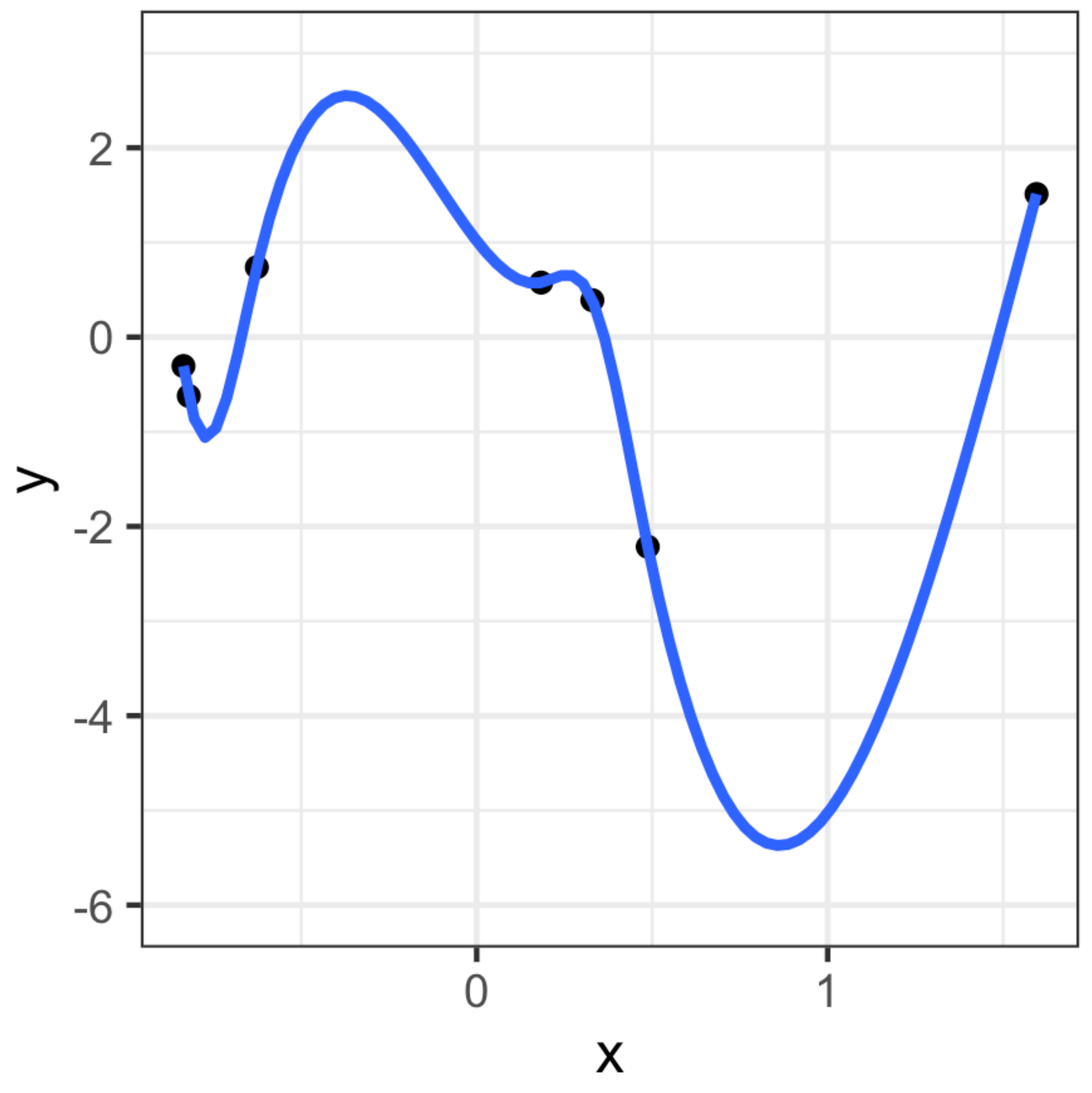
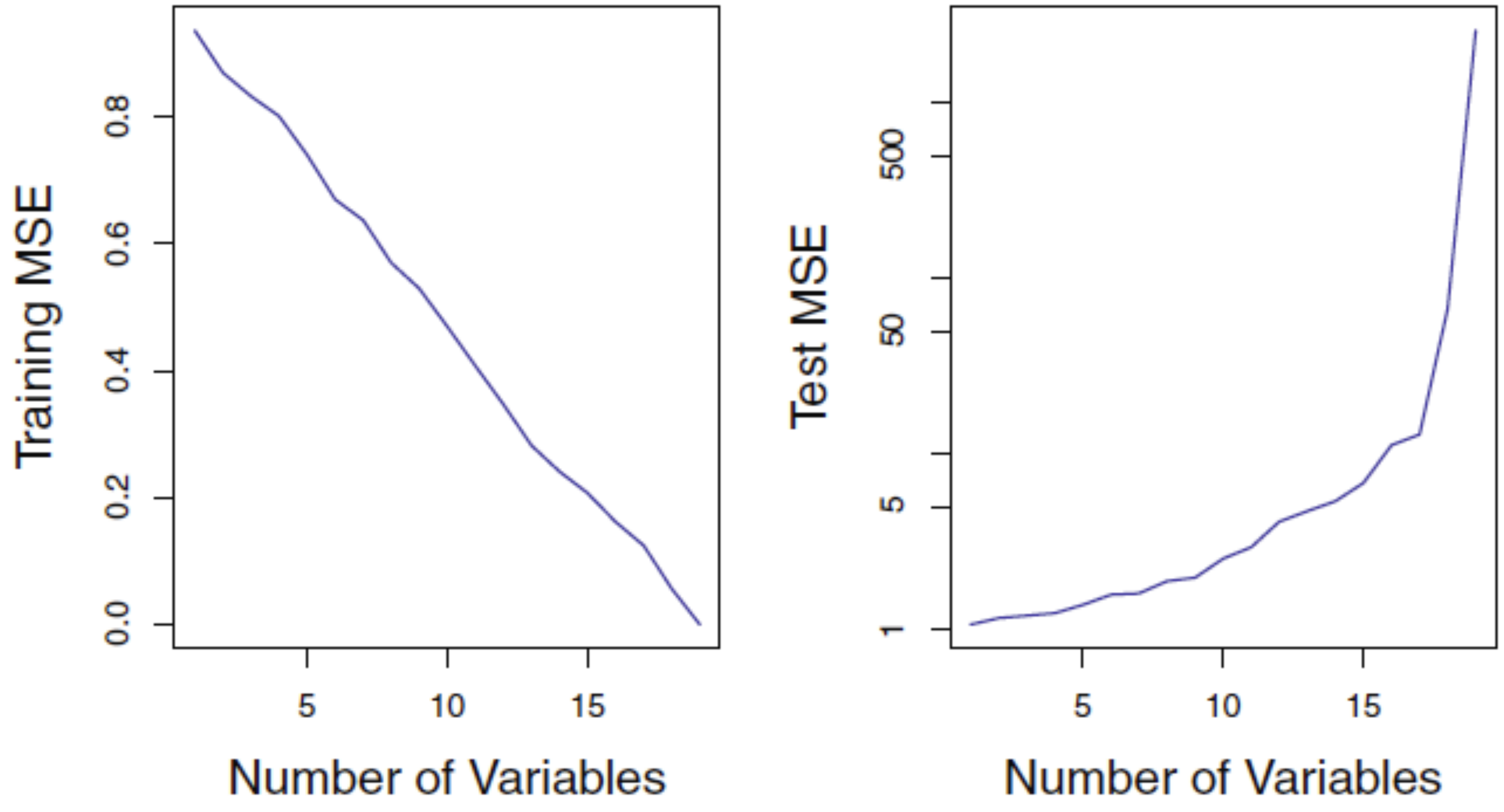
If $p < n$, recall that linear regression variance is $\sigma^2 p/n$.
Therefore, if $p \approx n$ then variance will be very high.



Linear models fit using too many features (i.e. too many degrees of freedom) perform poorly due to high variance.

Challenges in high dimensions (illustration)

Linear regression for $n = 20$; p features unrelated to response



The solution

The solution is to constrain the fitted coefficients in some way, e.g.:

1. Make sure fitted coefficients are not too large (ridge regression).

2. Make sure fitted coefficients are mostly equal to zero (lasso regression).

These constraints reduce the degrees of freedom of the fit, reducing variance.

We are still fitting p coefficients, but using fewer than p degrees of freedom.

Penalization: A way of constraining the fit

Recall least squares solution:

$$\widehat{\beta} = \underset{\beta_0,\beta_1,\ldots,\beta_{p-1}}{\arg\min} \sum_{i=1}^{n} (y_i - (\beta_0 + \beta_1 X_{i1} + \cdots + \beta_{p-1} X_{i,p-1}))^2.$$

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how well β fits the data ⟵ compromise ⟶ how wild β is

Example: L0-penalized regression

Consider the penalized regression

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with $P(\beta) = |\{j : \beta_j \neq 0\}|$.

The L0 penalty P counts the number of nonzero entries in β, and creates sparse solutions $\widehat{\beta}$.

The optimization above is computationally infeasible, so in practice we use a different penalty (called the lasso) to achieve sparsity (stay tuned for Lecture 4).

How and when penalization works

Penalization reduces the variance, but increases the bias of the predictions.

$\Rightarrow$ Reduces test error when reduction in variance outweighs increase in bias.

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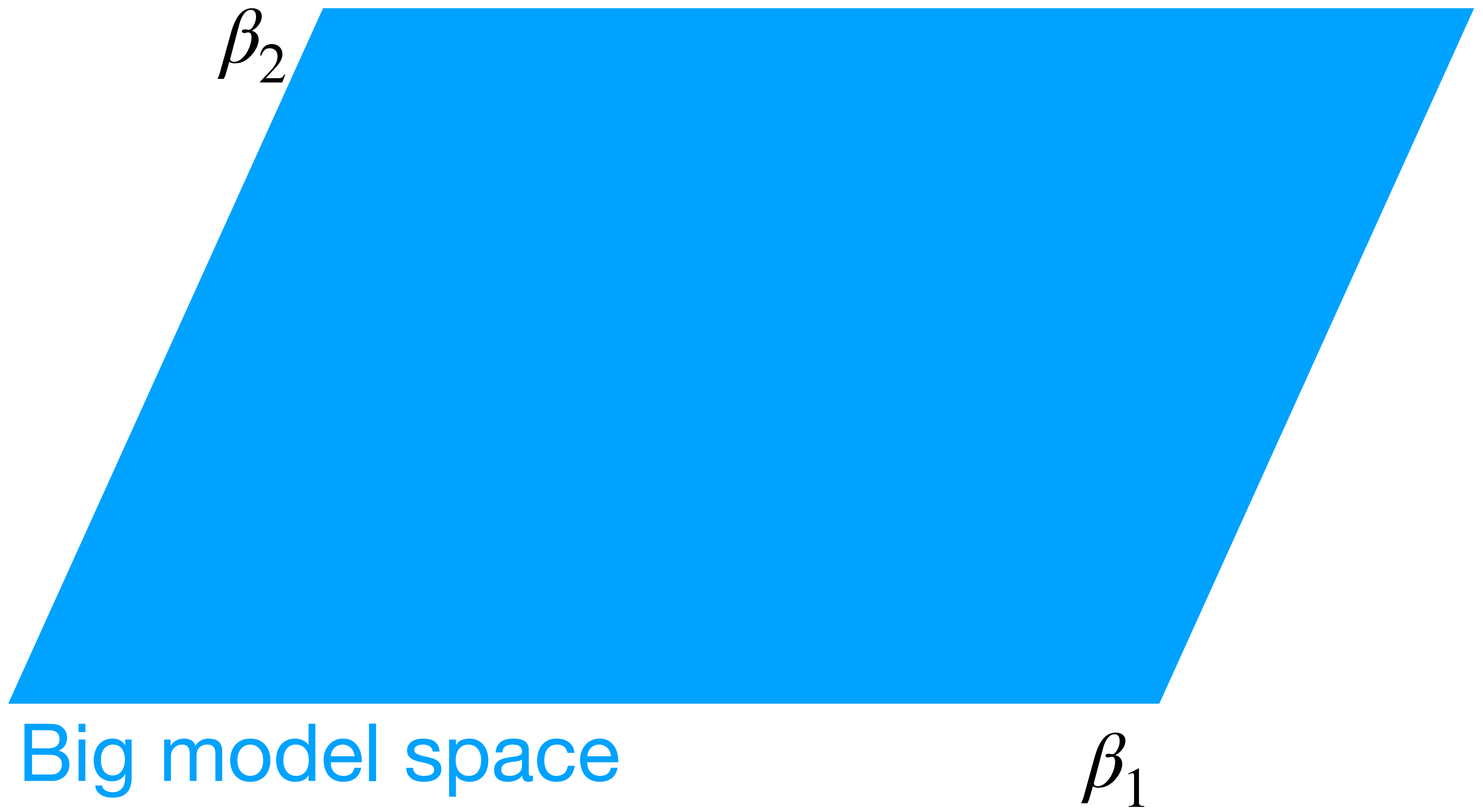

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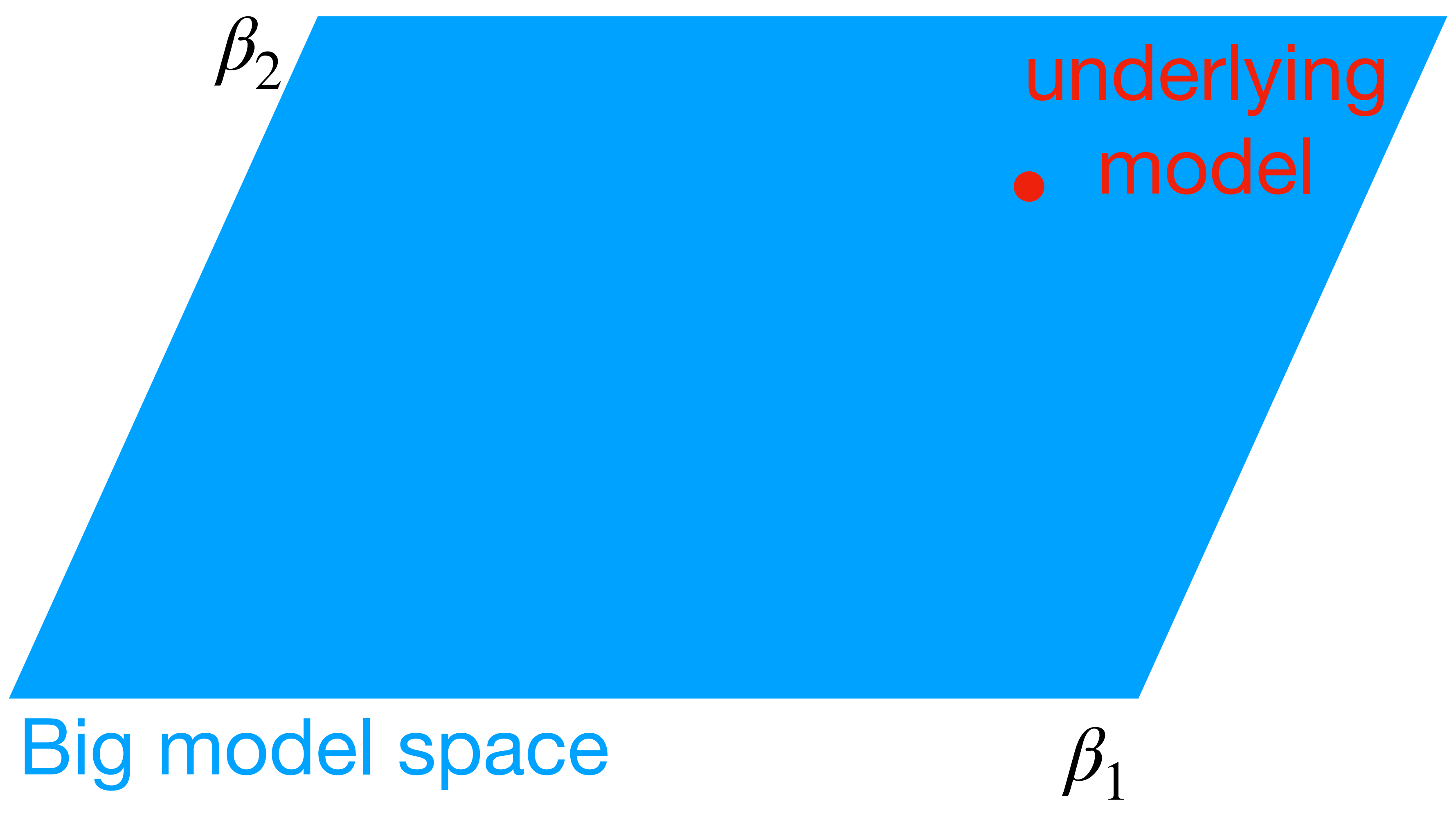

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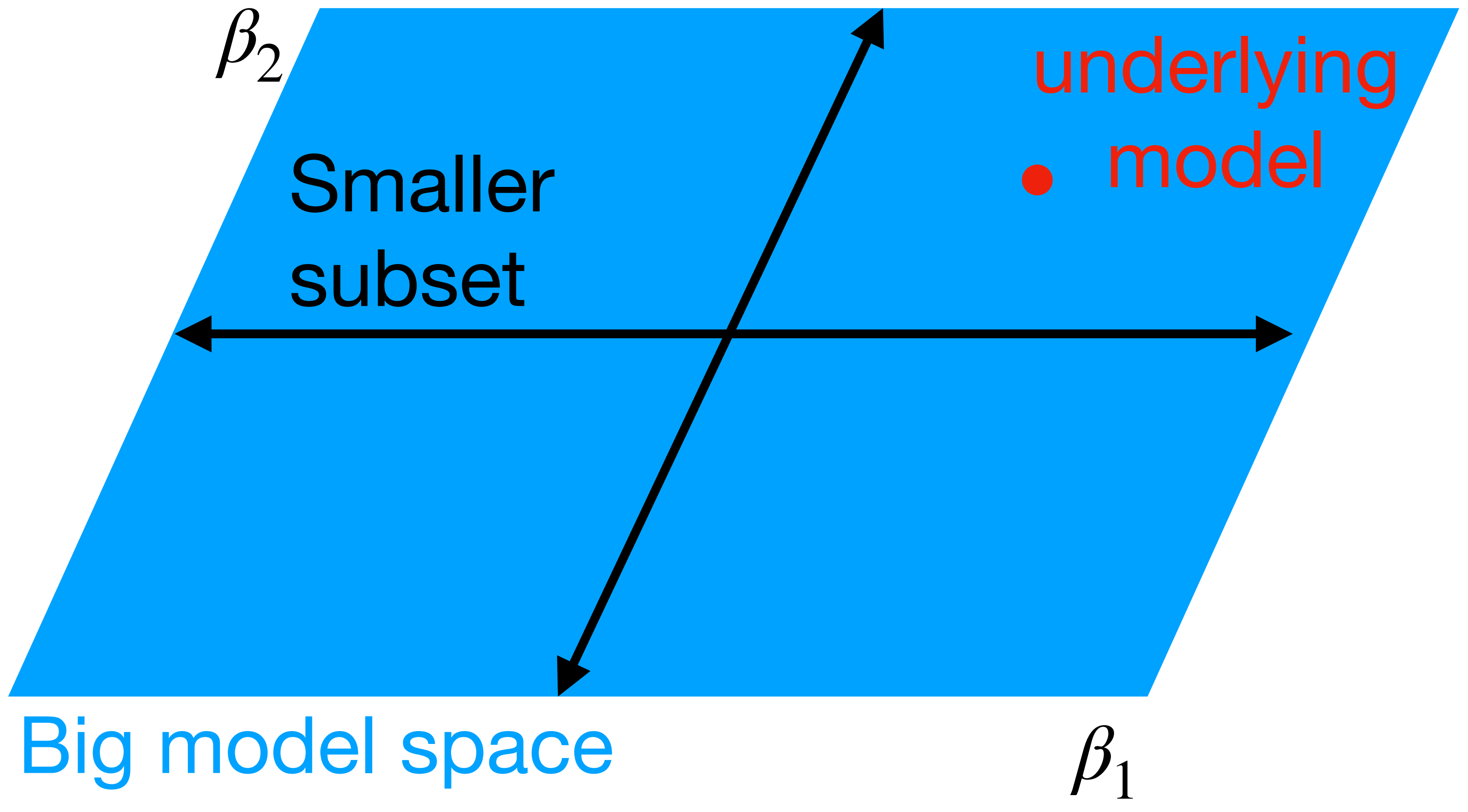

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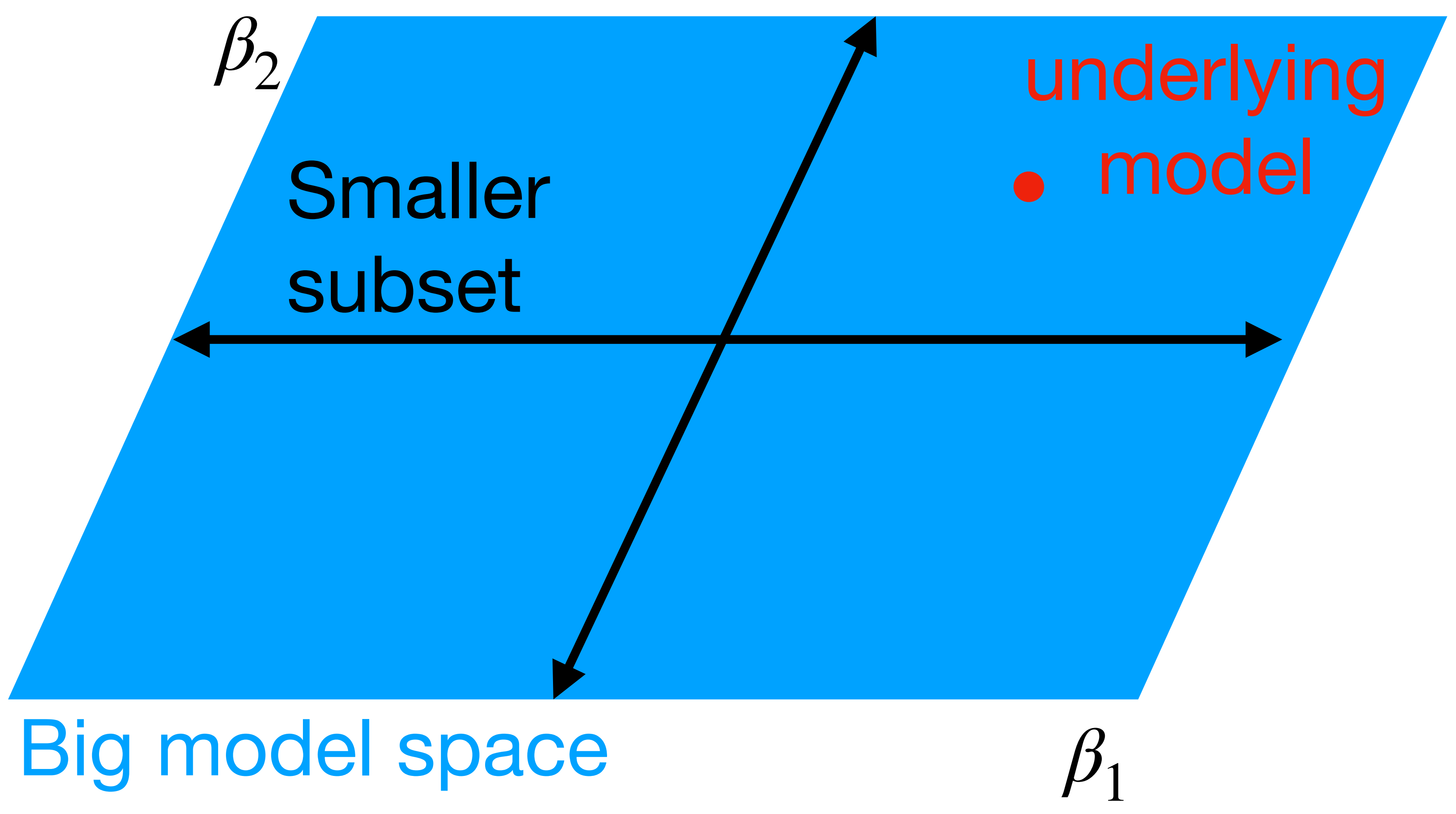

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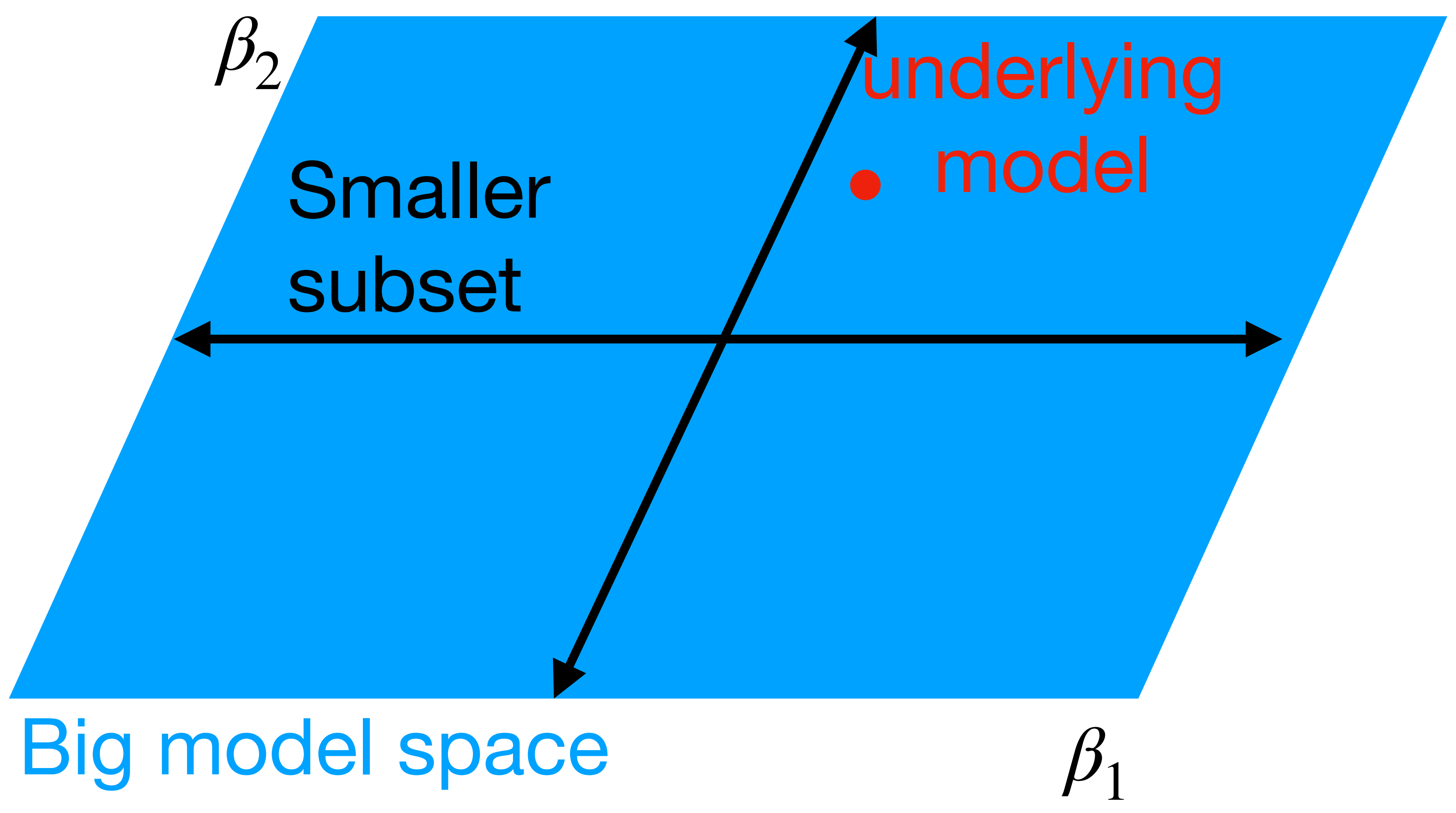

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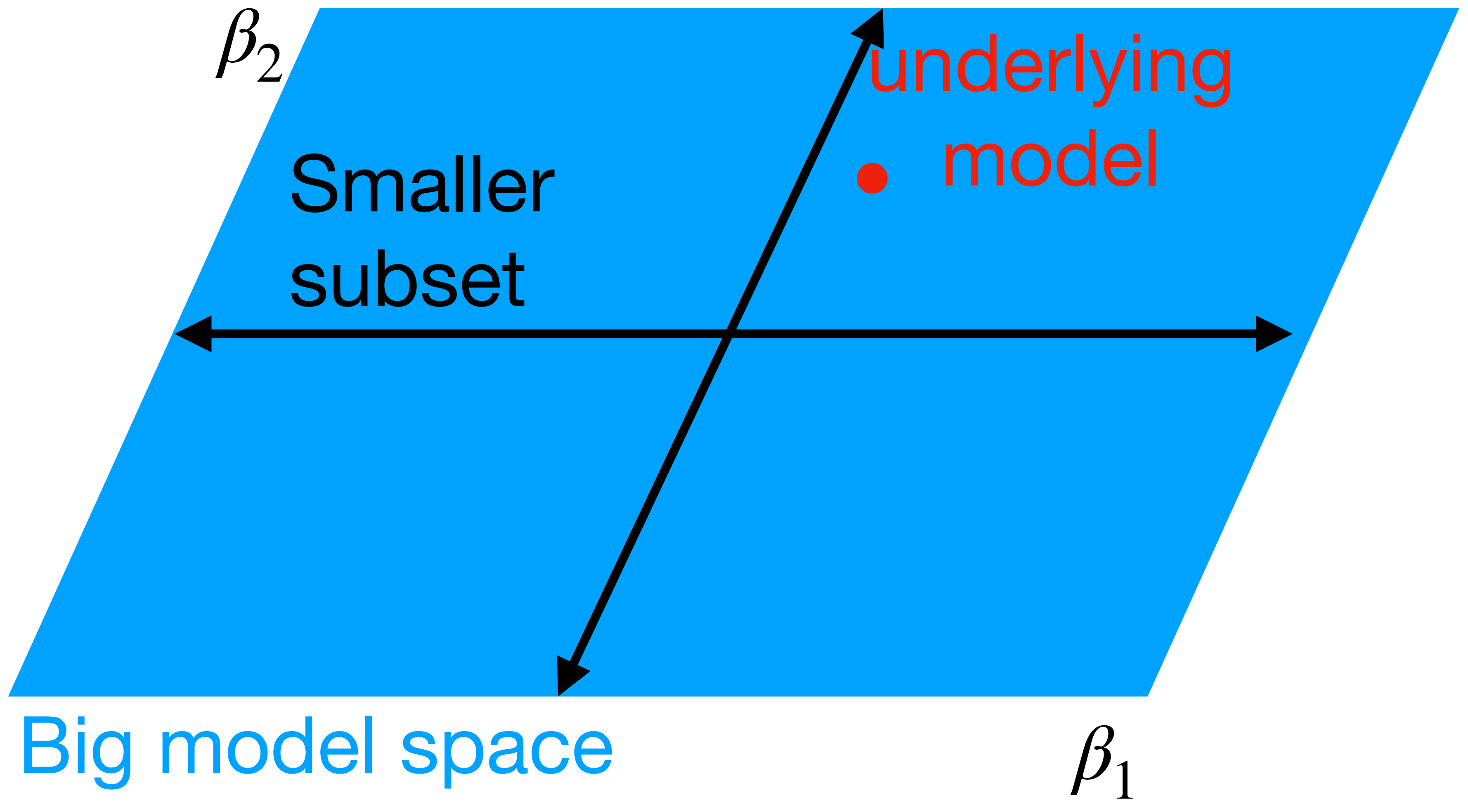

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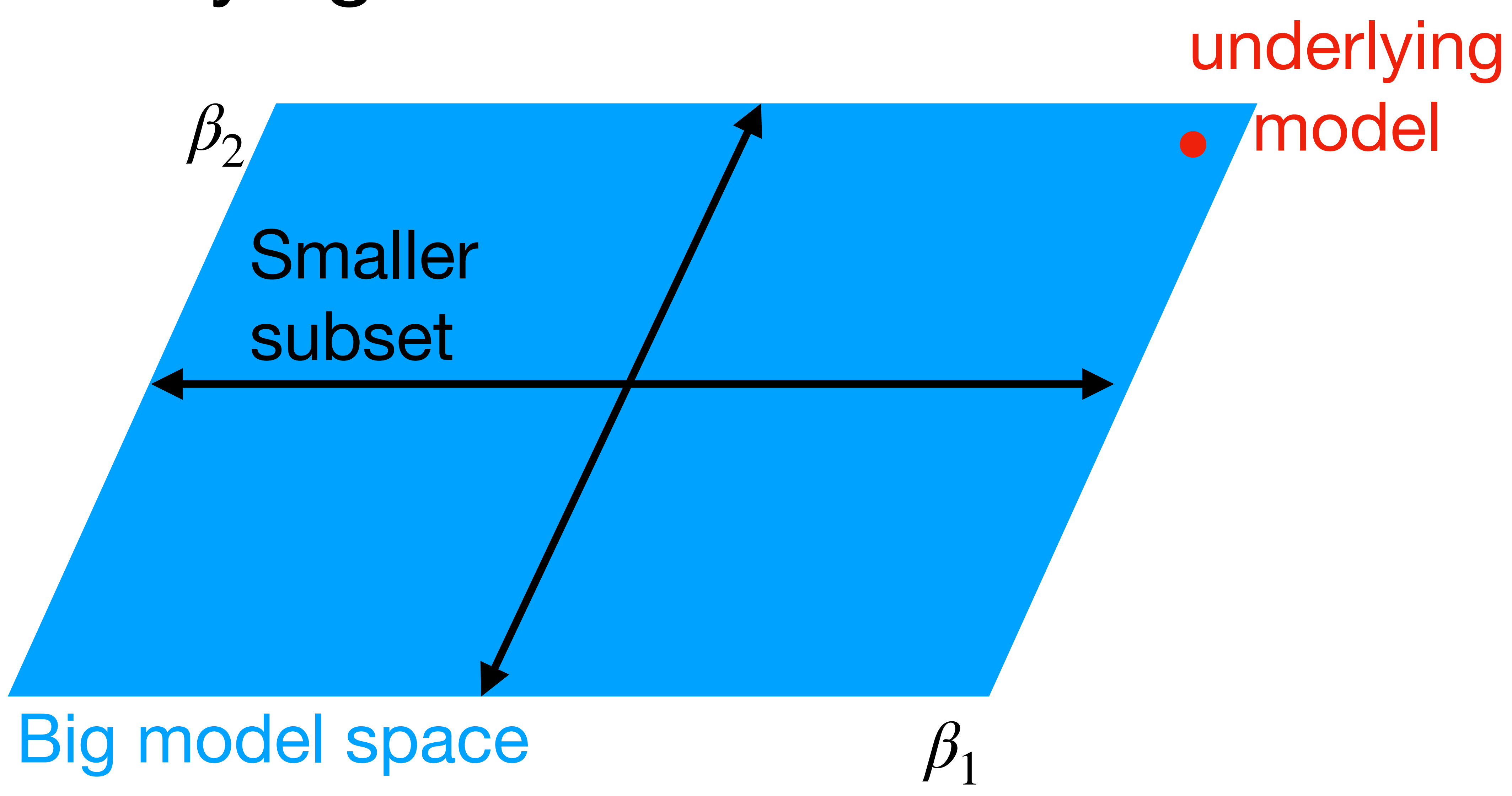

Looking ahead to lectures 3 and 4

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Quiz Practice