Linear and logistic regression

STAT 4710

October 3, 2023

Rolling into Unit 3

- ✓ **Unit 1:** R for data mining
- ✓ **Unit 2:** Prediction fundamentals
- **Unit 3:** Regression-based methods
- **Unit 4:** Tree-based methods
- **Unit 5:** Deep learning

Lecture 1: Linear and logistic regression

Lecture 2: Regression in high dimensions

Lecture 3: Ridge regression

Lecture 4: Lasso regression

Lecture 5: Unit review and quiz in class

Predicting a response based on multiple features

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$$Y = \beta_0 + \beta_1 X_1 + \cdots + \beta_{p-1} X_{p-1} + \epsilon.$$

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Let's review:

- Continuous and categorical features in linear models
- Interpretation of linear regression coefficients
- How to fit a linear regression model

Continuous and categorical features in linear regression

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Example 2 (binary feature): $X_2 =$ sex. It does not make sense to write $\beta_2 X_2$; what does $3 \times$ "male" mean? Instead, use dummy coding: $X_2 = I(\text{sex} = \text{male})$.

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Example 3 (categorical feature): $X_3 =$ education. It does not make sense to write $\beta_3 X_3$. Instead, map education onto multiple dummy variables:
$X_3 = I(\text{education} = \text{high school})$, $X_4 = I(\text{education} = \text{"college"})$, etc.

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To avoid redundancy, use dummy variables for all levels except one baseline.

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Note: Linear regression coefficients do not necessarily imply causation.

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The least squares optimization problem can be solved in closed form.

What if the response is binary?

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> Default
# A tibble: 10,000 x 4
   default student balance income
   <fct>   <fct>     <dbl>  <dbl>
 1 No      No         730. 44362.
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We build a model to approximate

$$\mathbb{P}[\text{default} = \text{Yes} \mid \text{student}, \text{balance}, \text{income}]$$

and then predict

$$\text{default} = \begin{cases} \text{Yes}, & \text{if } \widehat{\mathbb{P}}[\text{default}] \geq 0.5; \\ \text{No}, & \text{if } \widehat{\mathbb{P}}[\text{default}] < 0.5. \end{cases}$$

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How do we model probability of default?

Options for modeling probability of default

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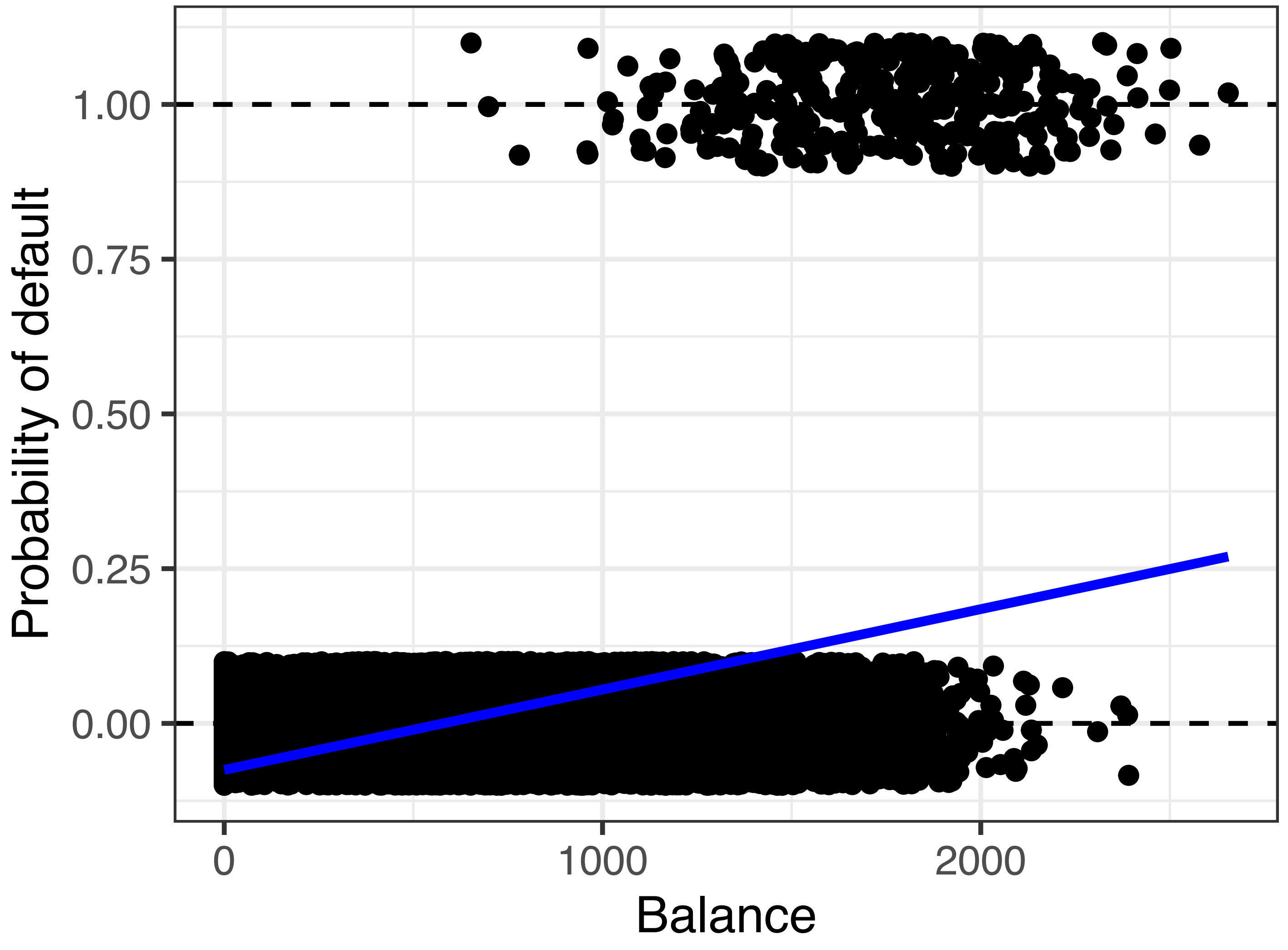
Linear regression

$\beta_0 + \beta_1 \cdot \text{balance}$



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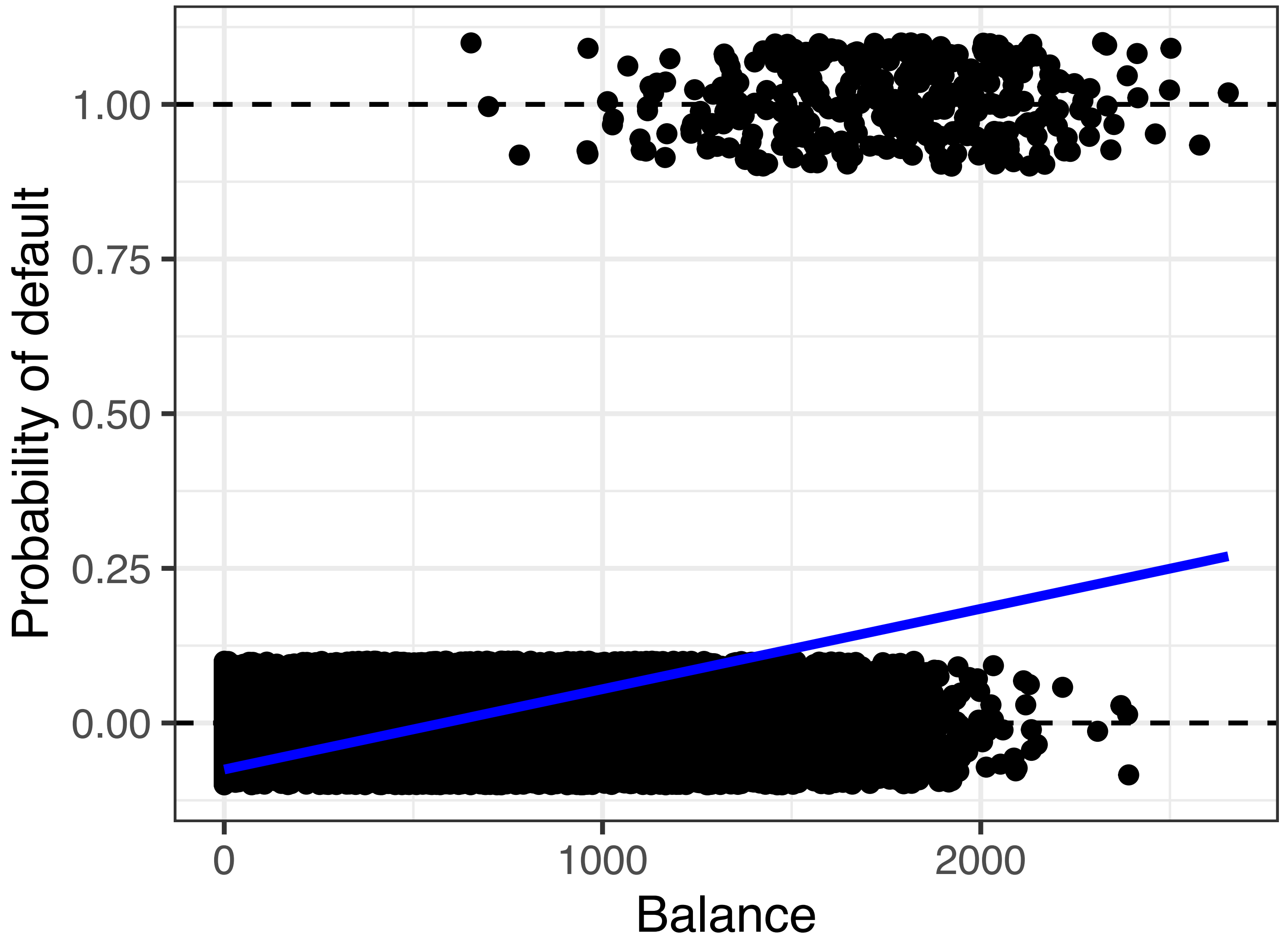

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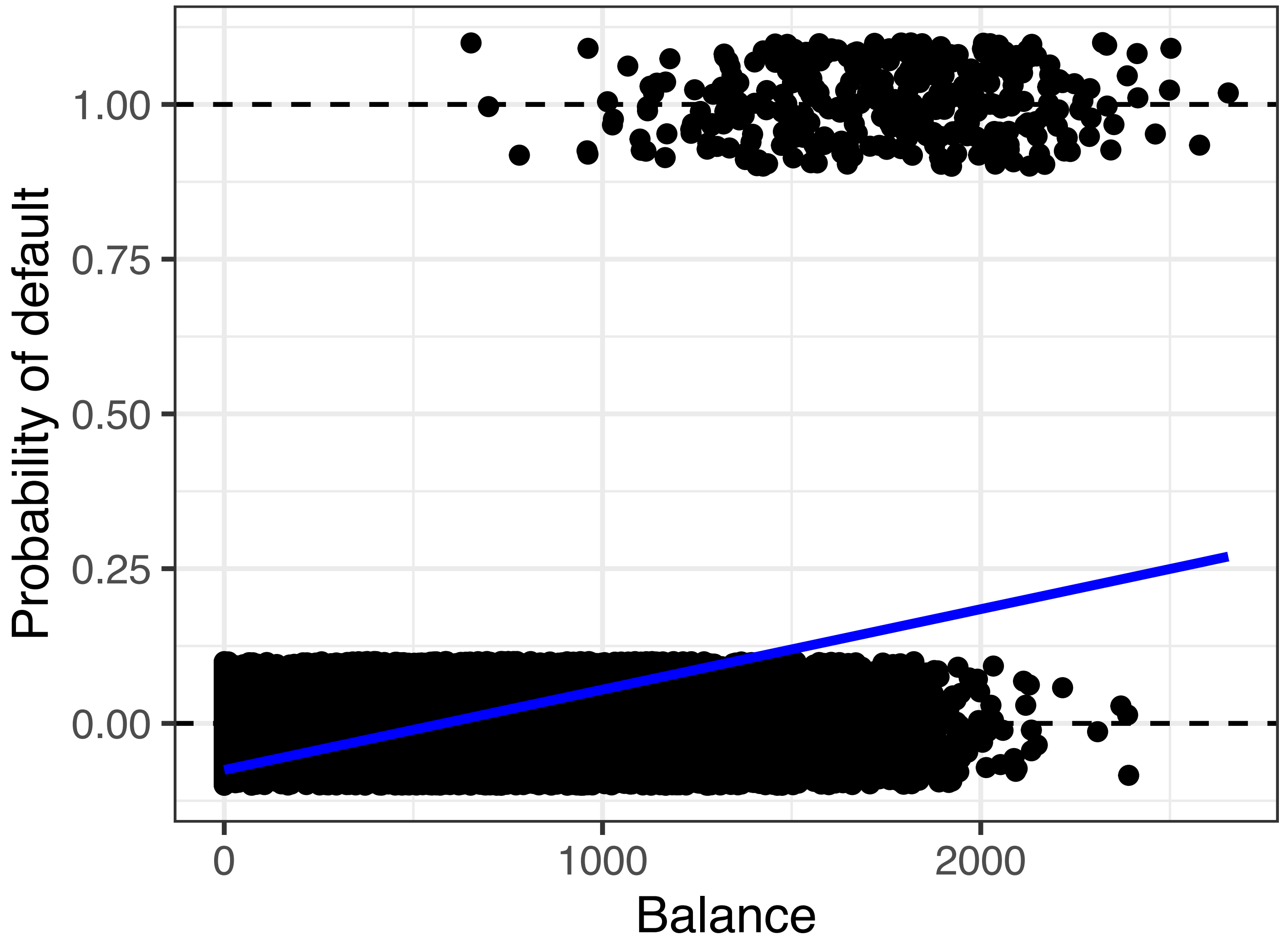

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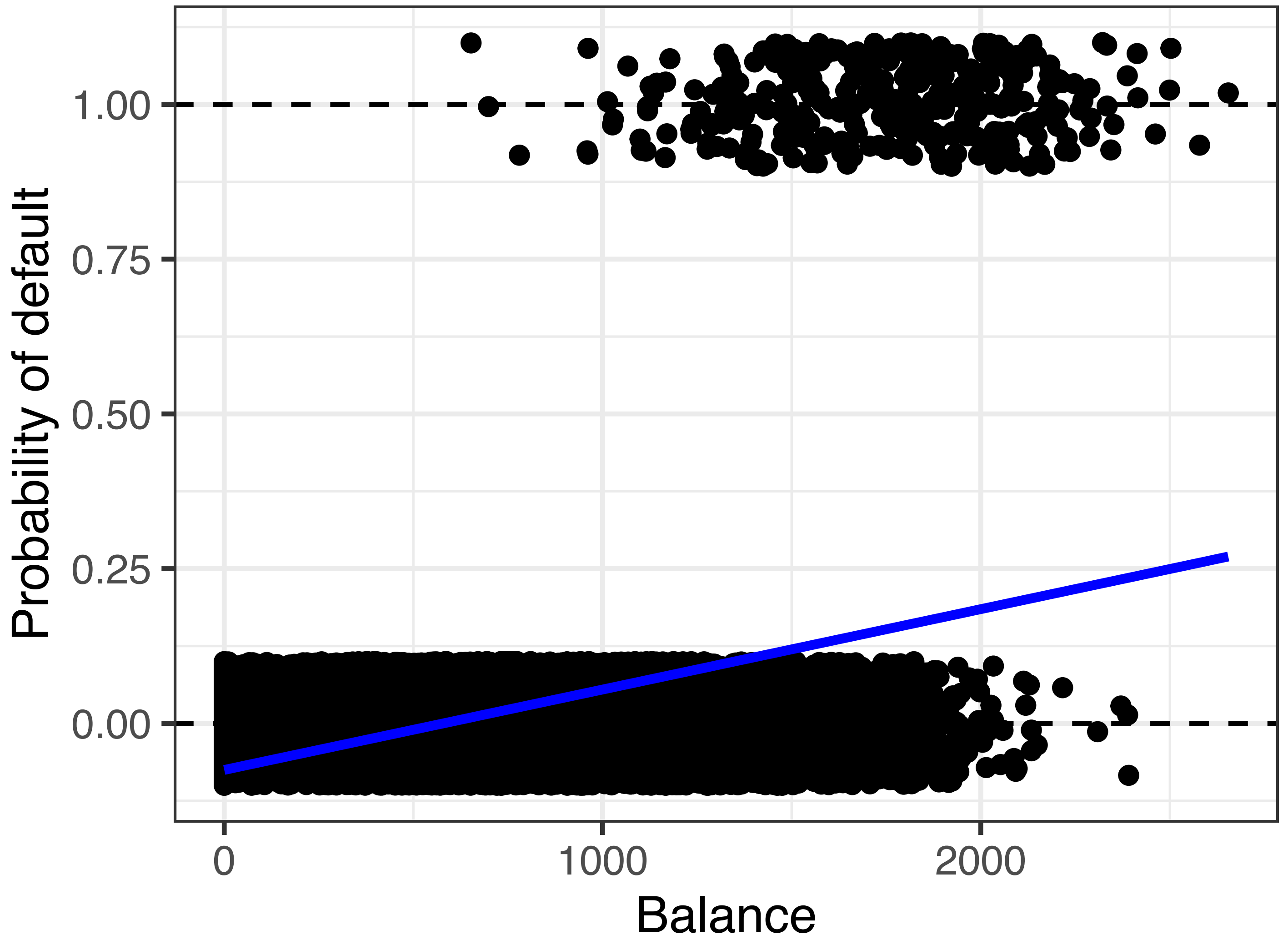
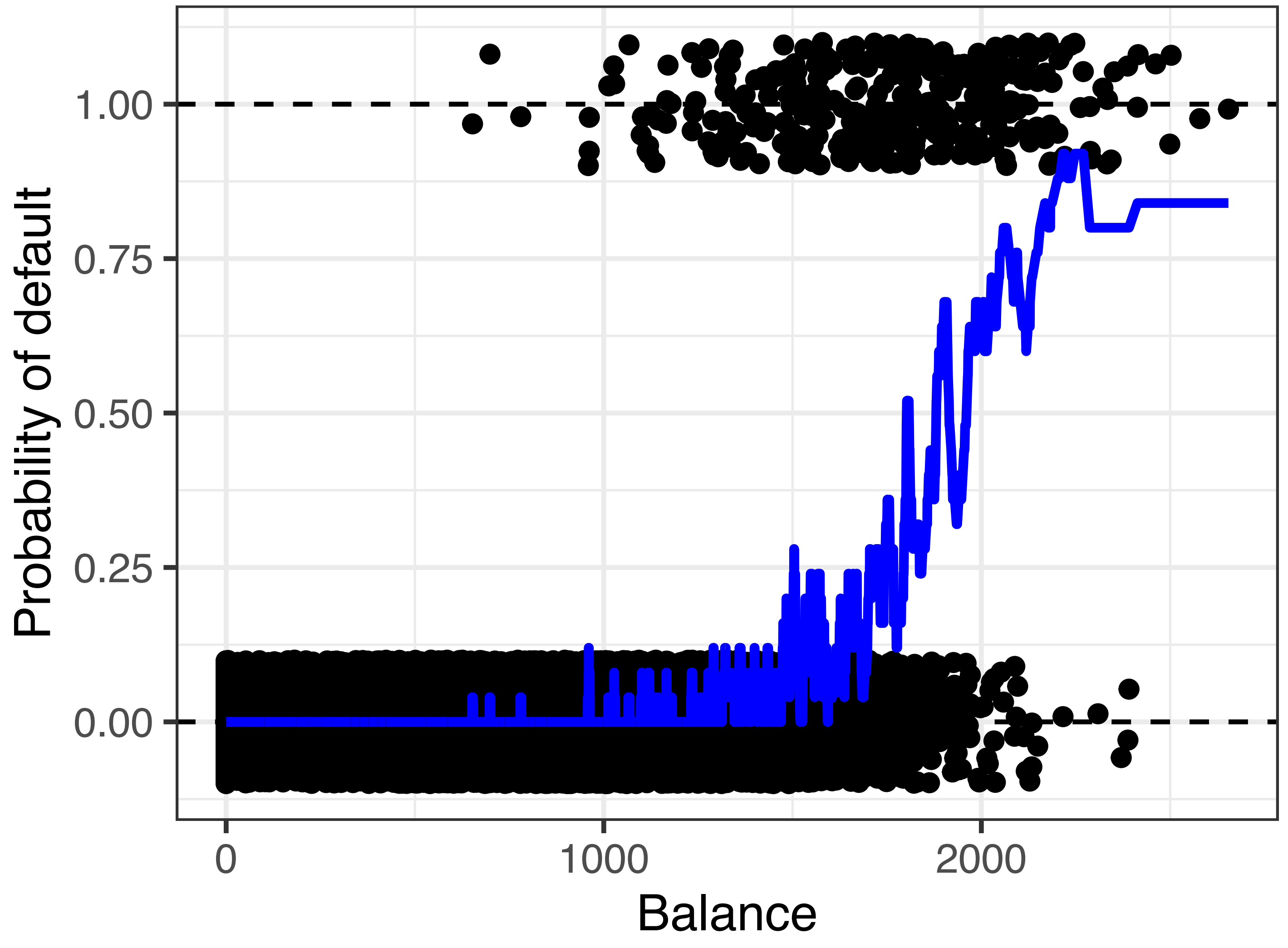


K-nearest neighbors

proportion of K N. N. who defaulted



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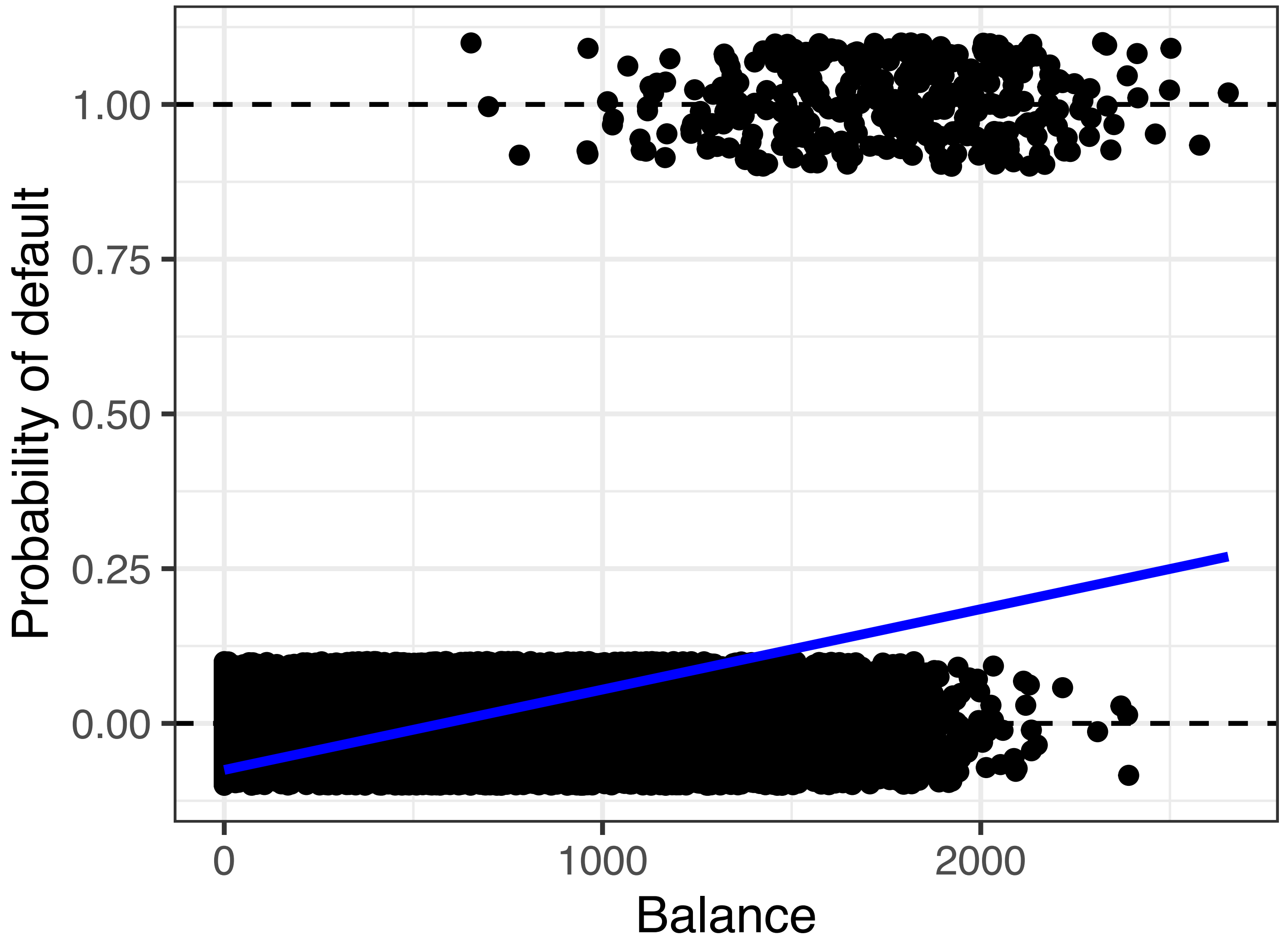

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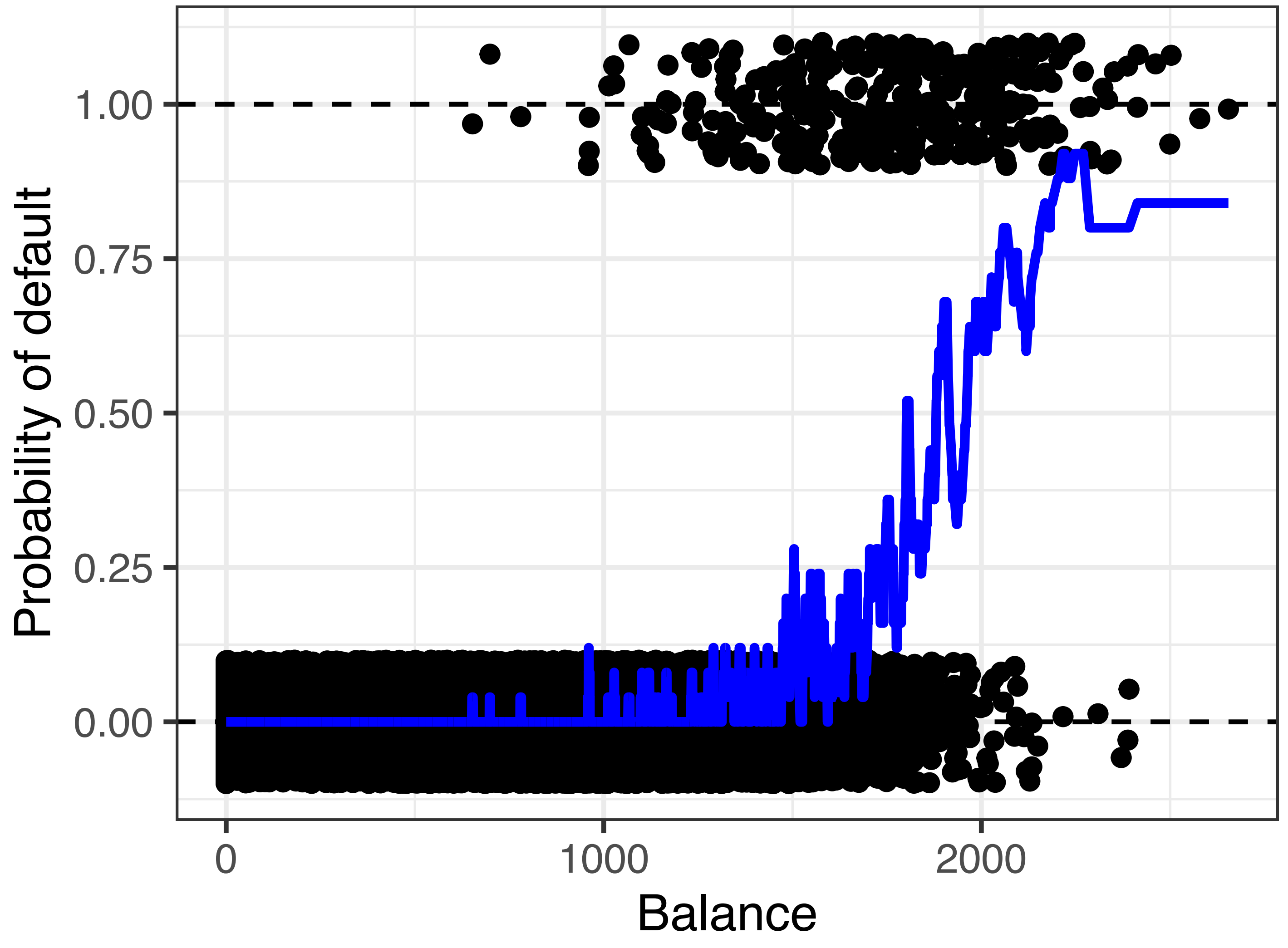

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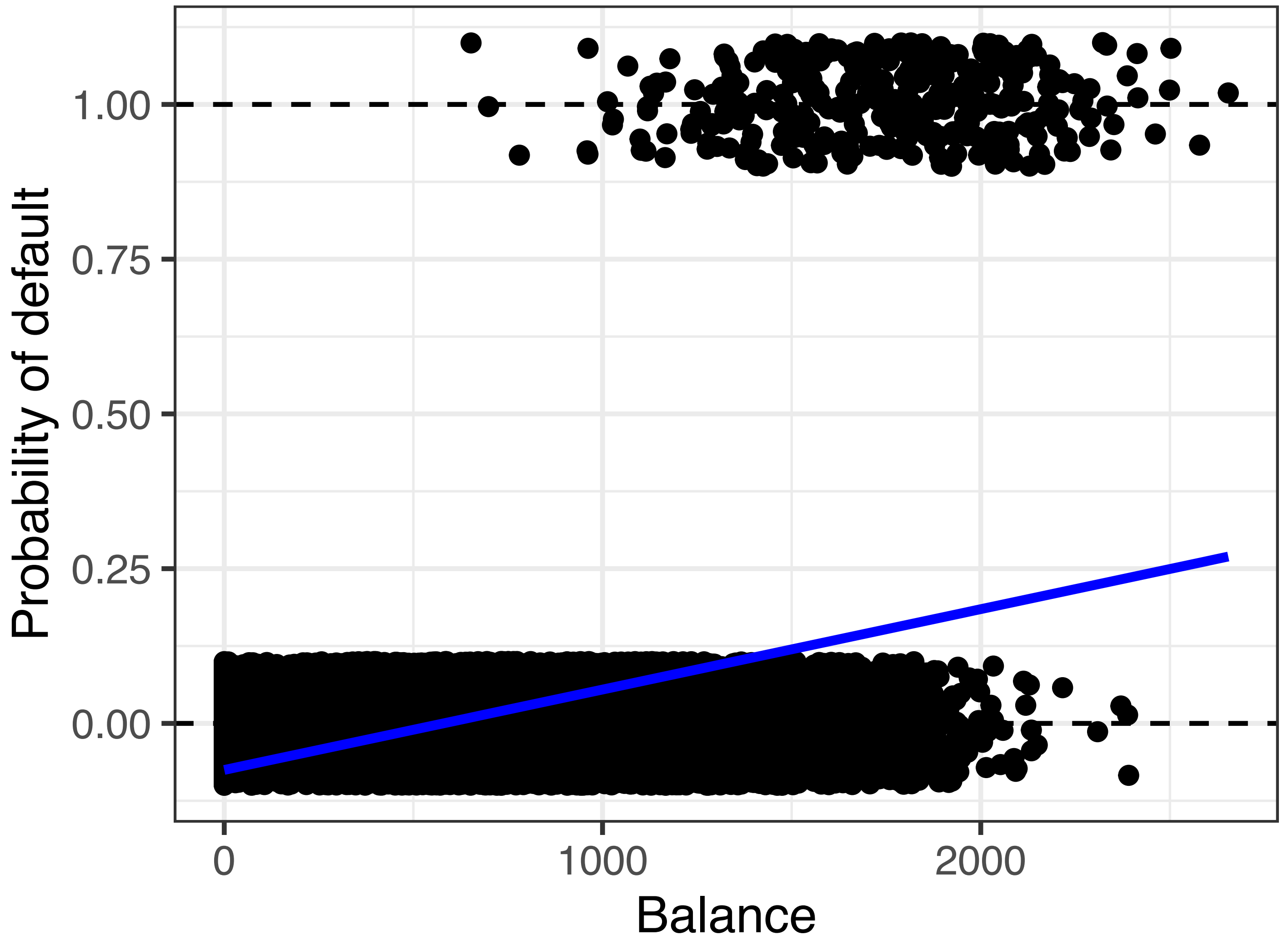

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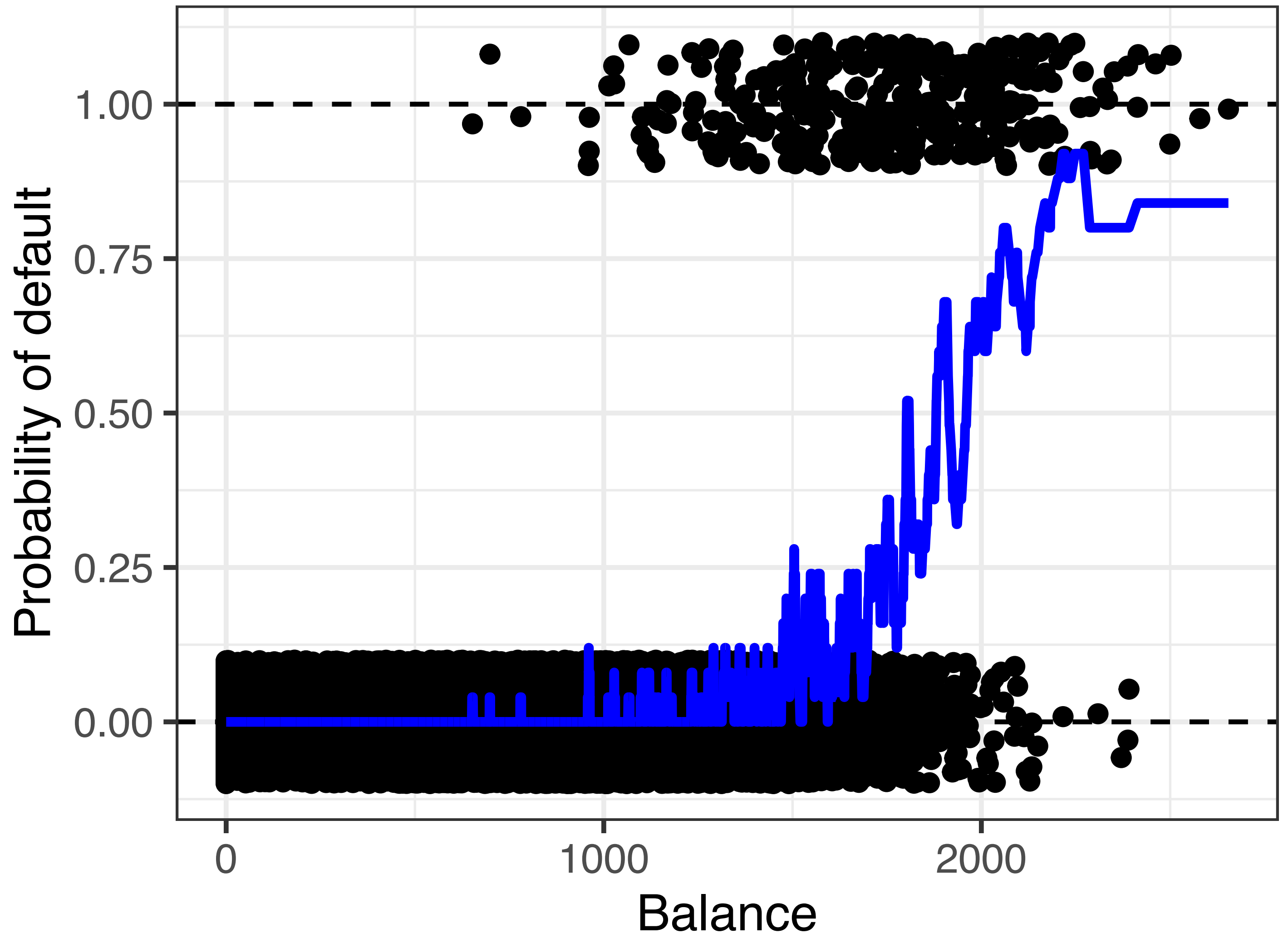

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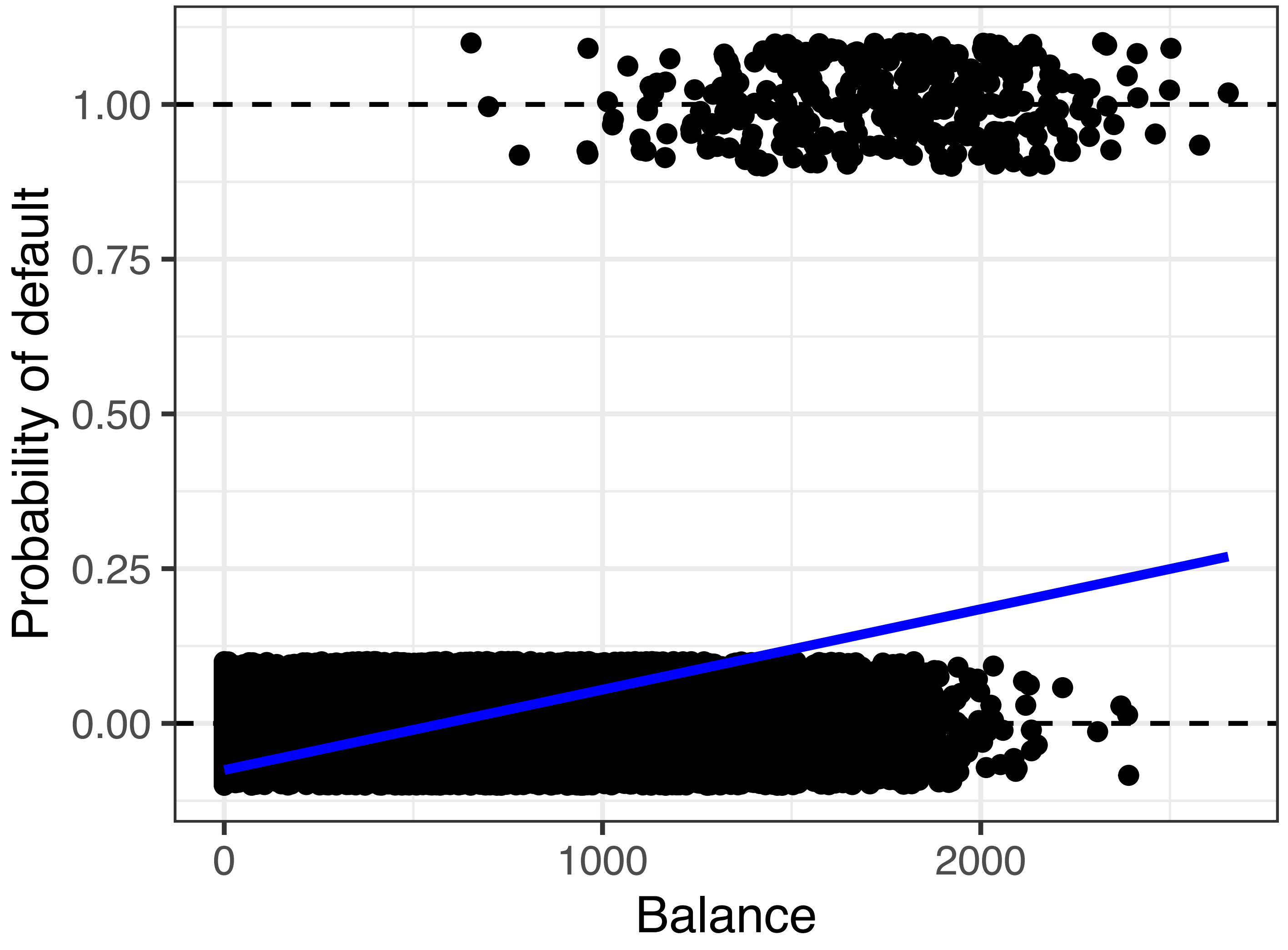

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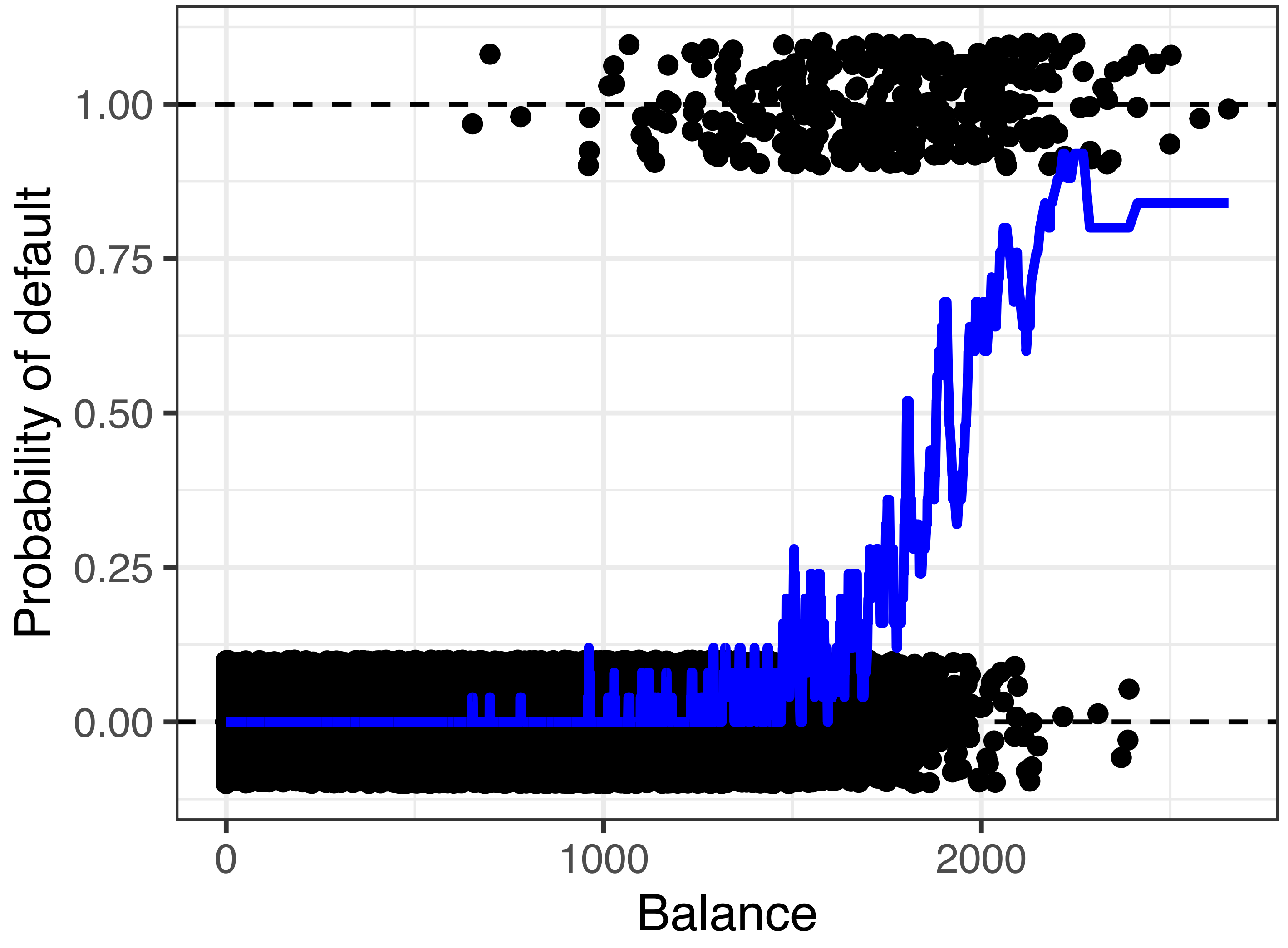
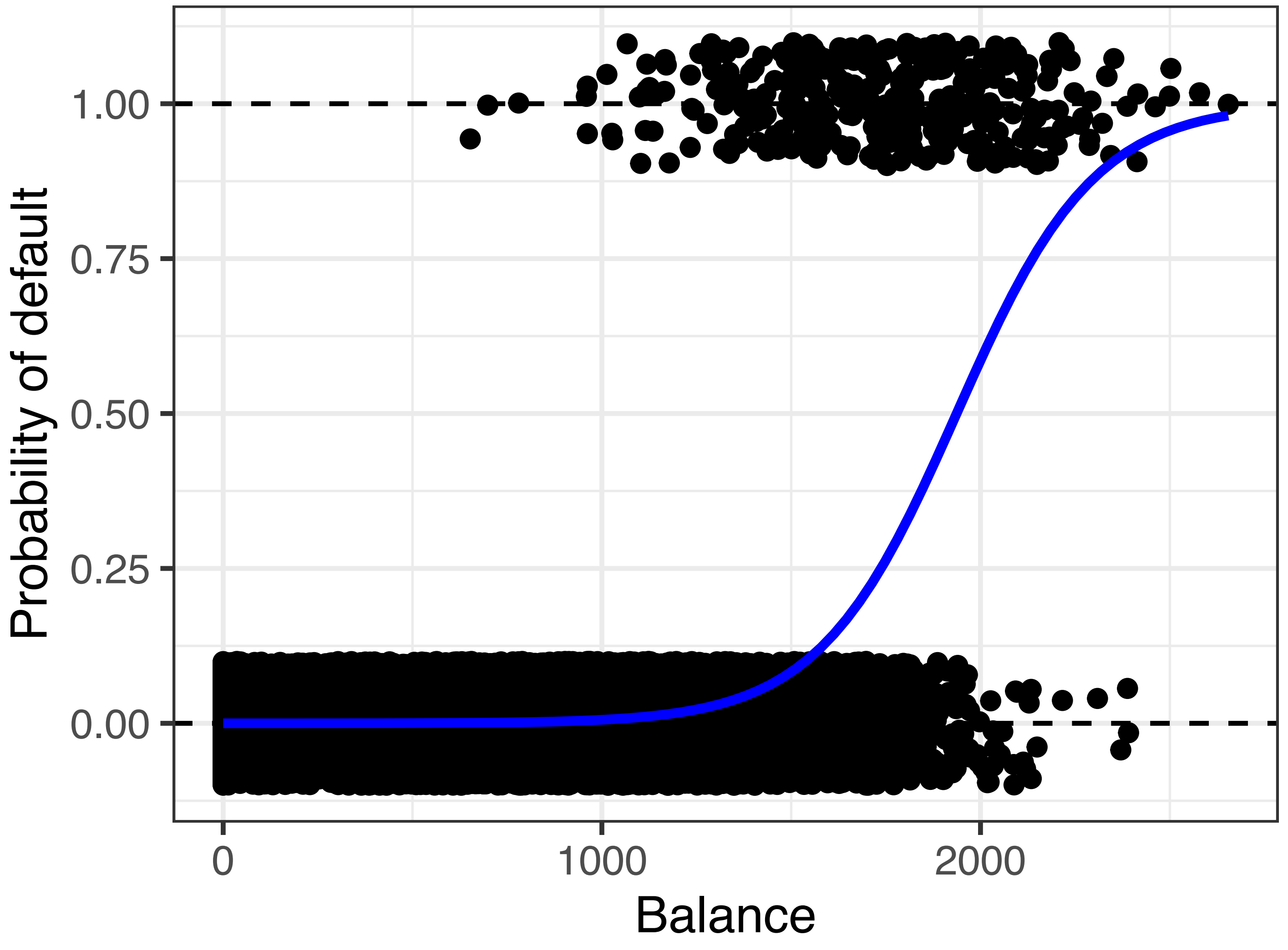


Logistic regression

$\text{logistic}(\beta_0 + \beta_1 \cdot \text{balance})$



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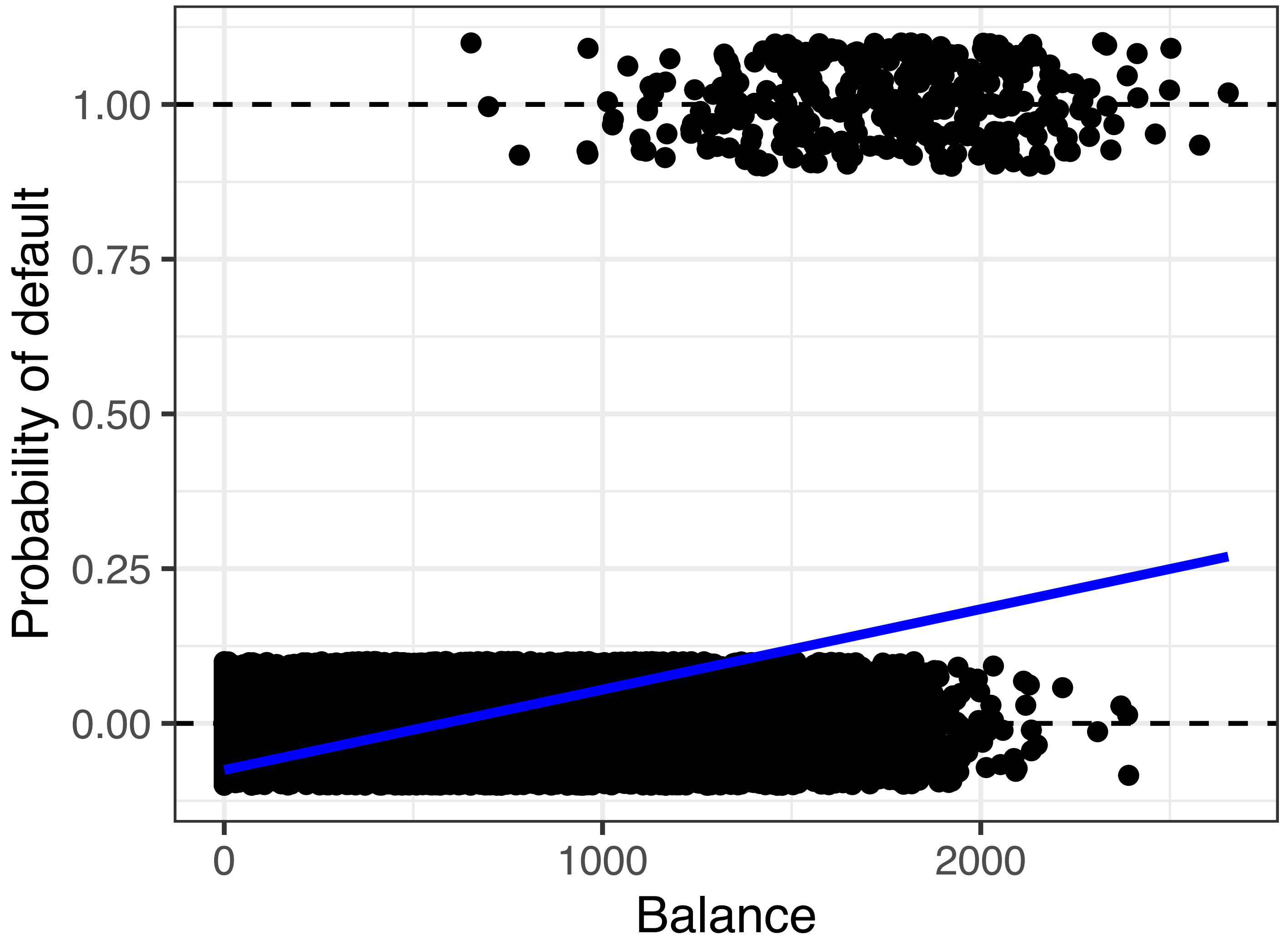

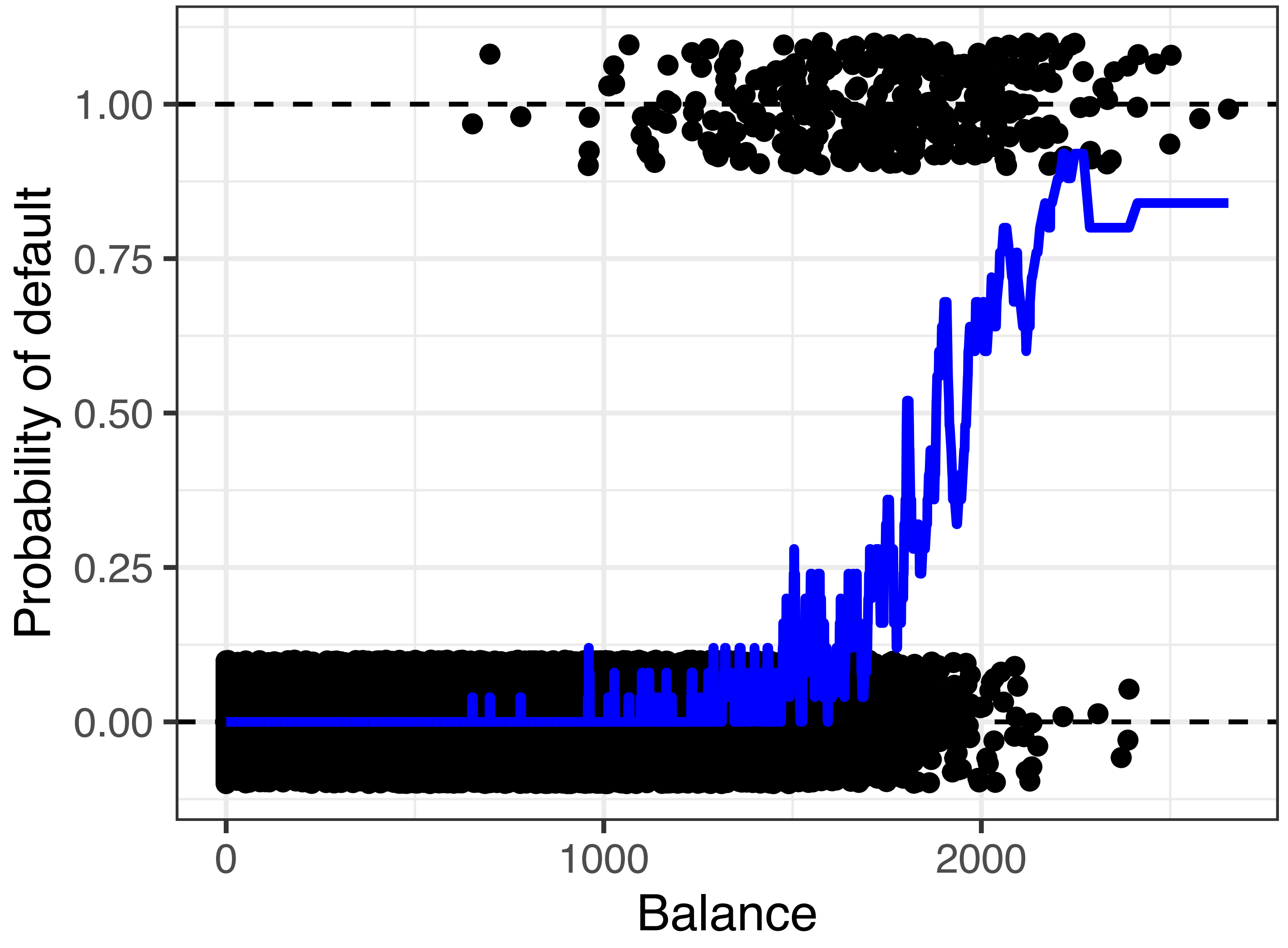

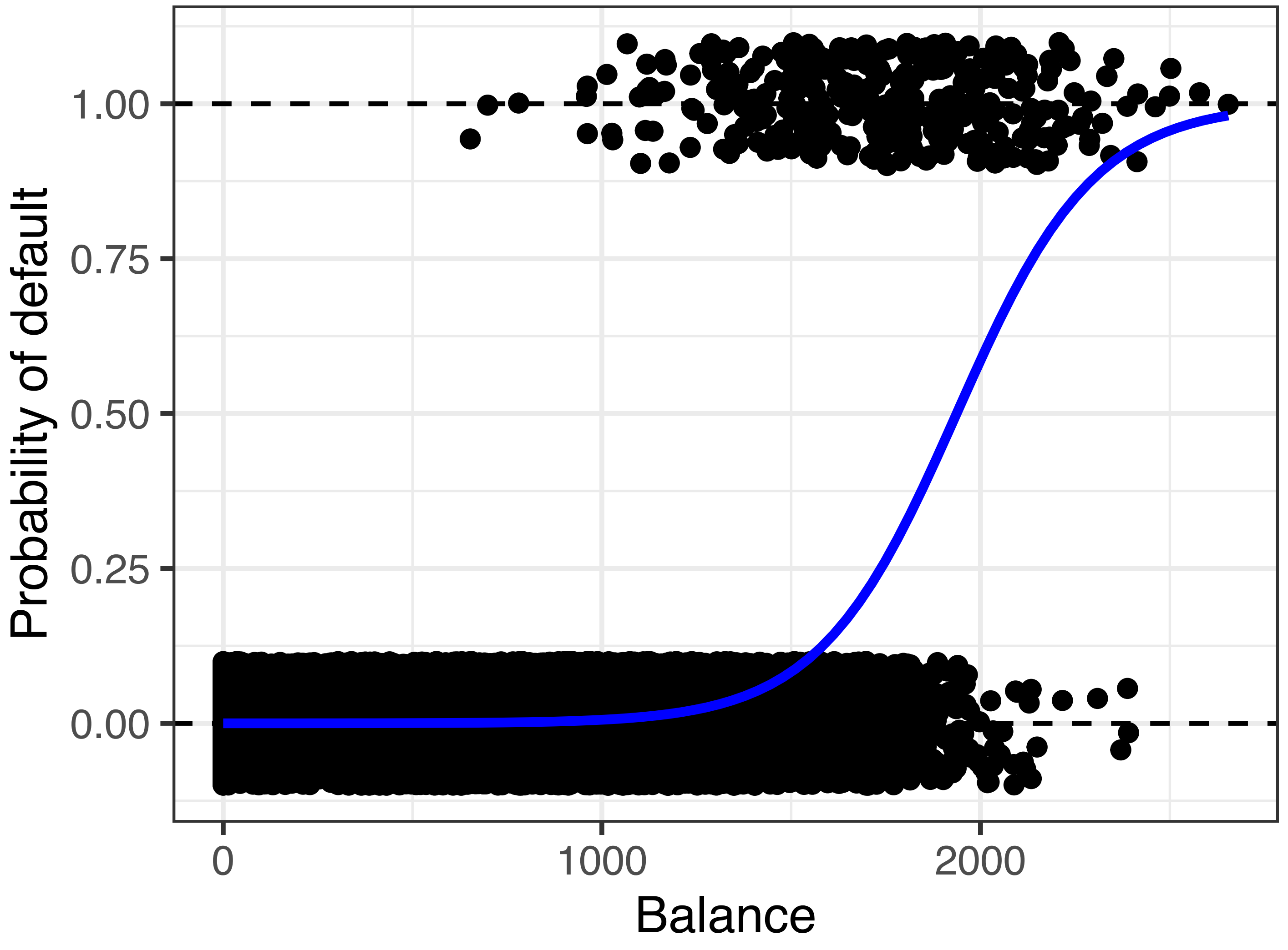

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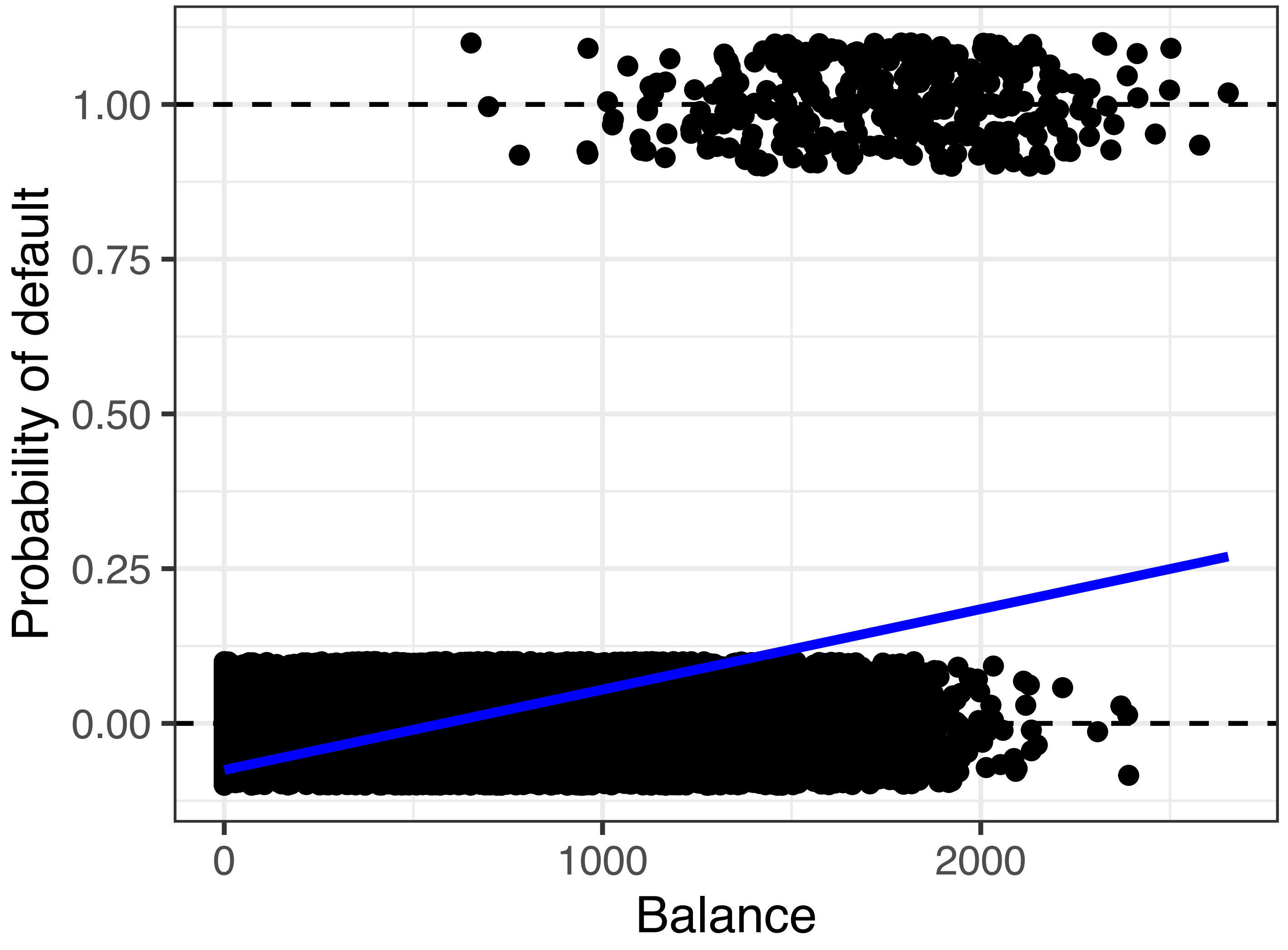

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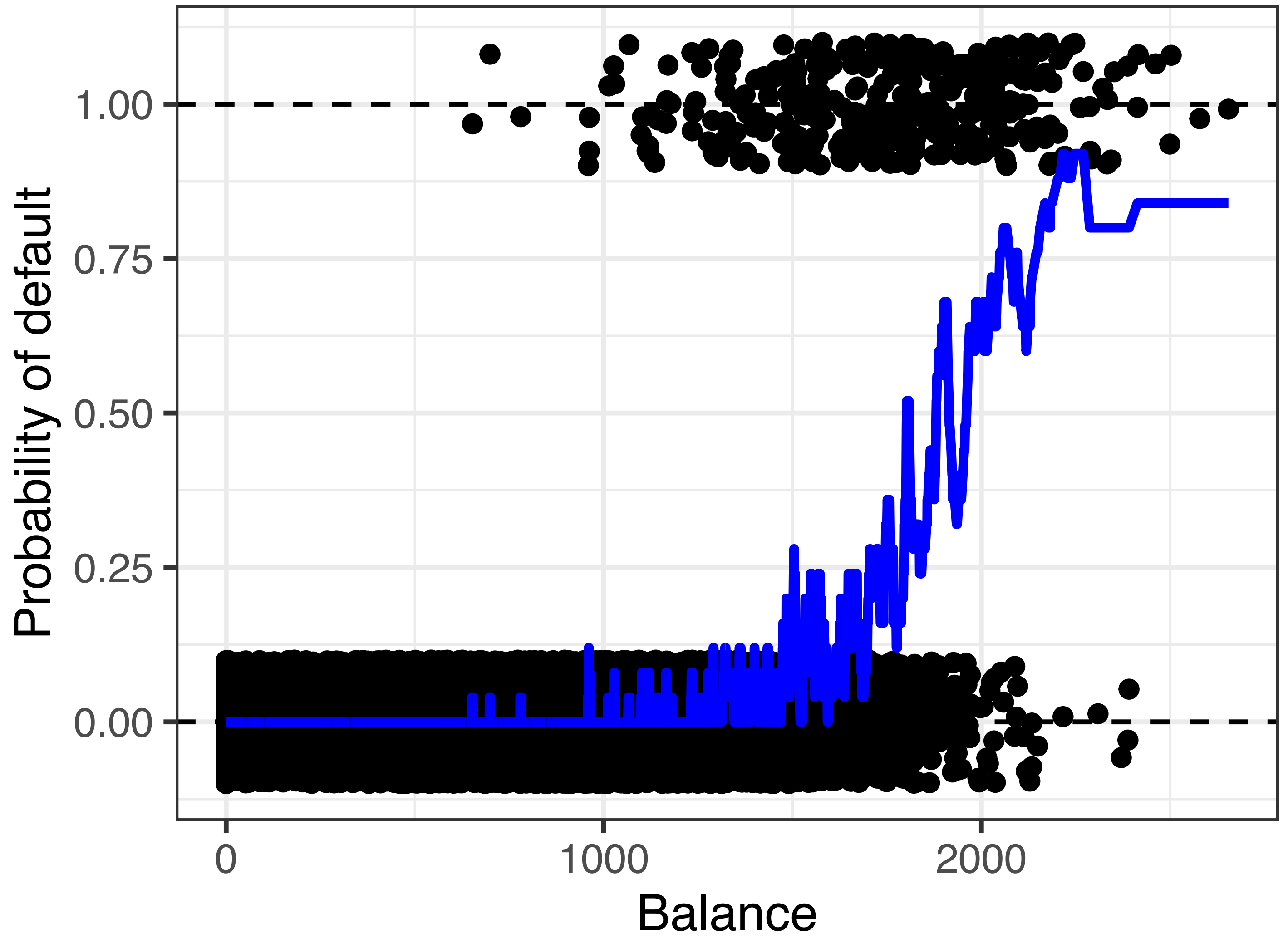

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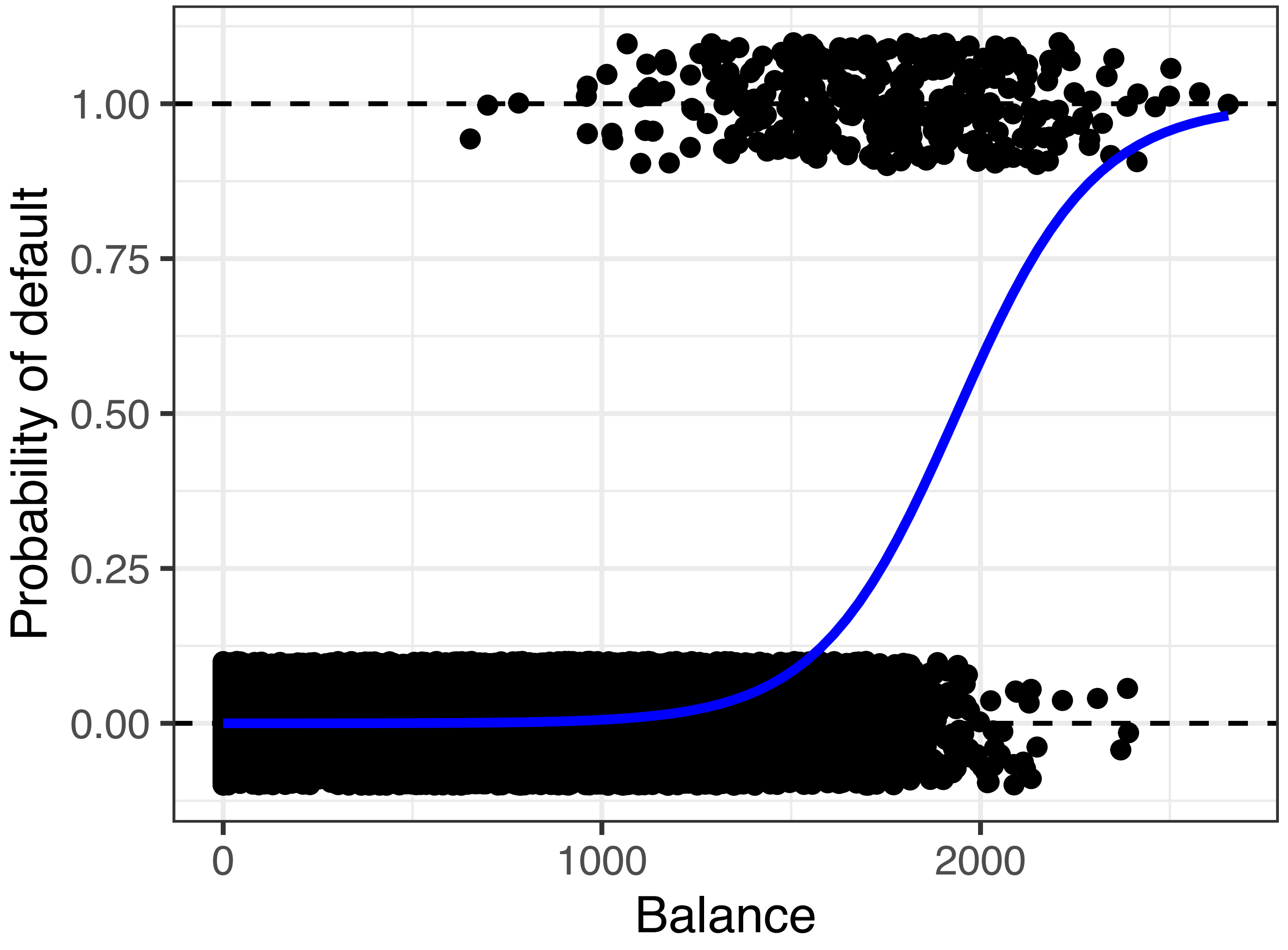

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The logistic regression model

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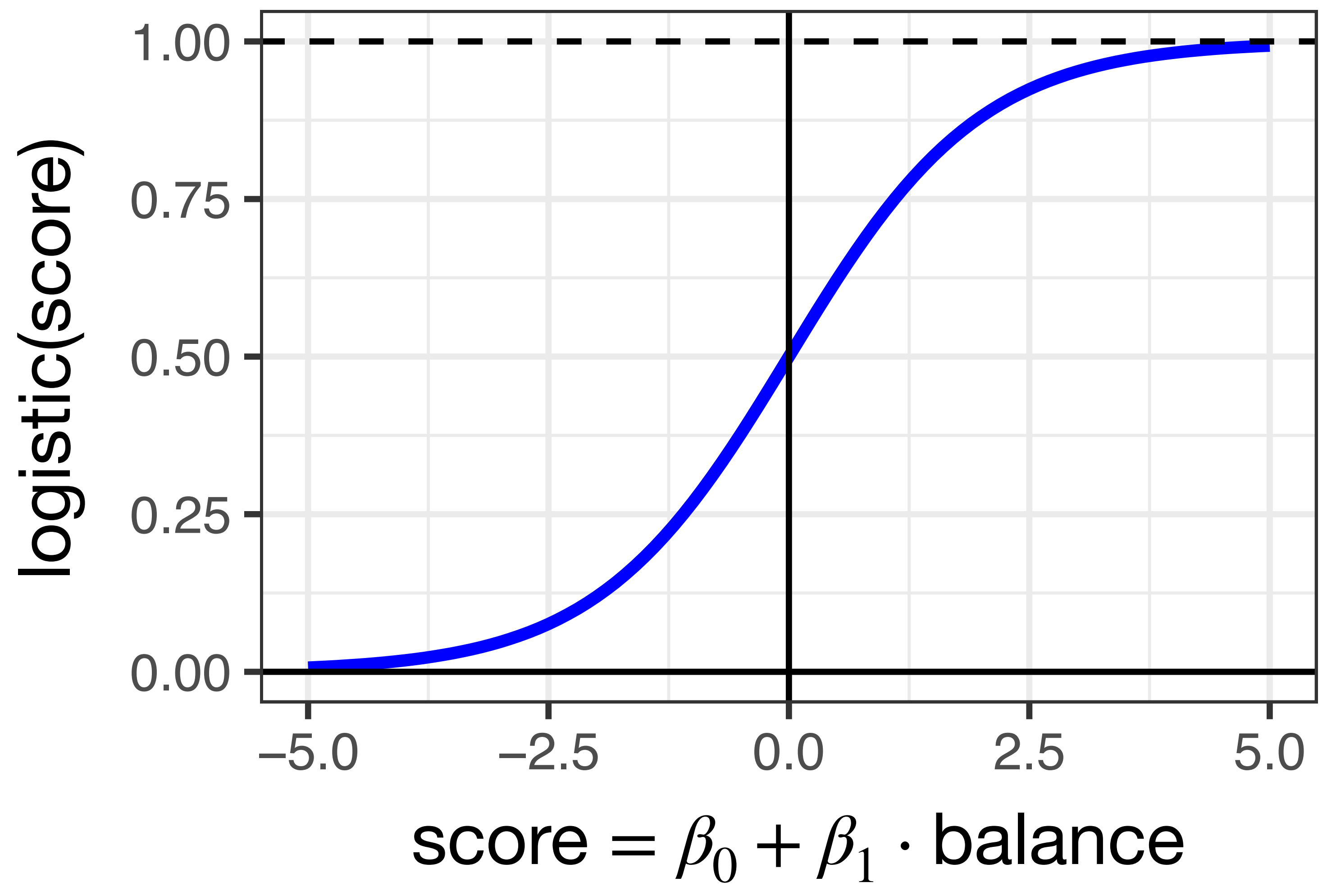

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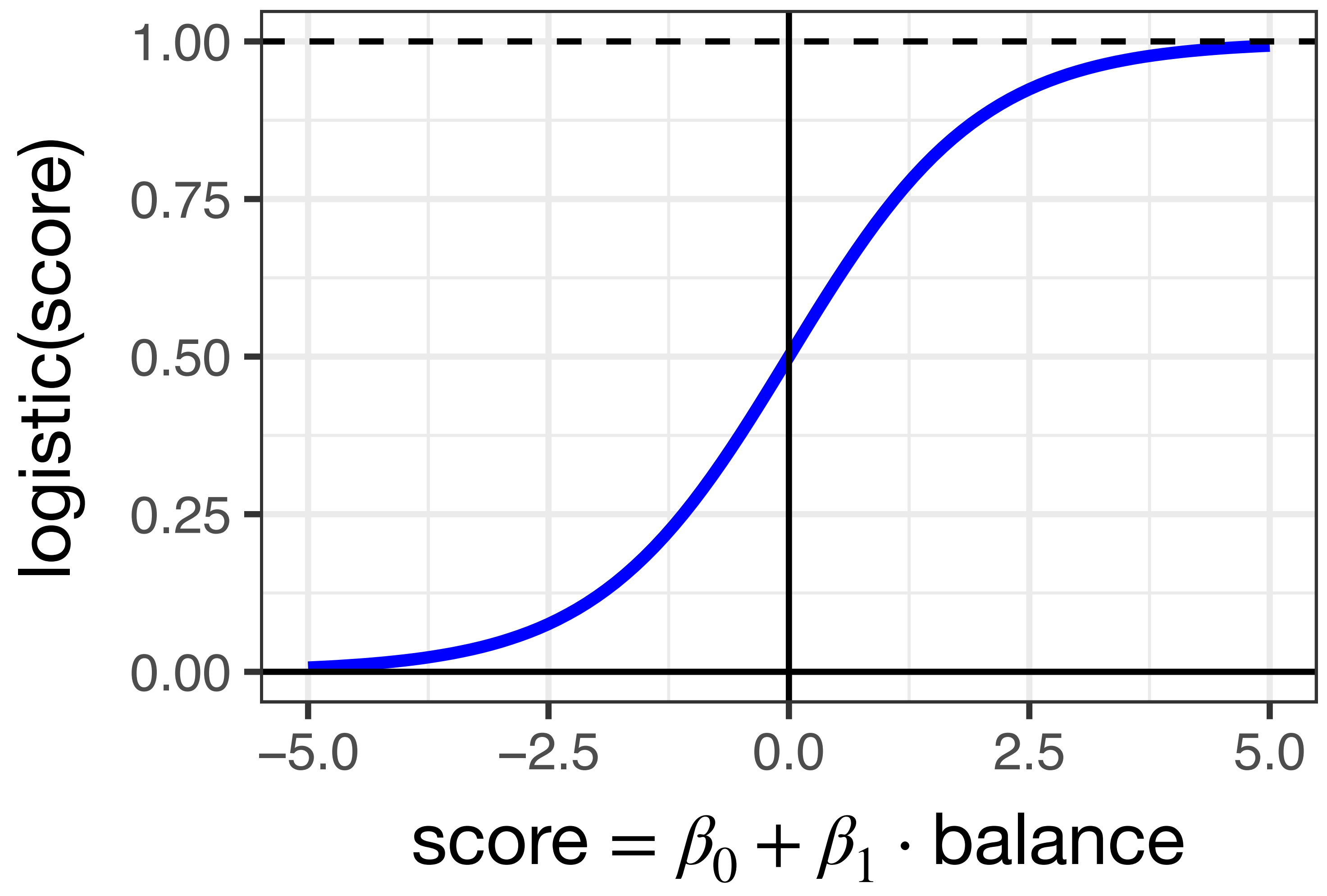

Logistic regression model:

$$\mathbb{P}[\text{default} \mid \text{balance}] = \text{logistic}(\beta_0 + \beta_1 \cdot \text{balance})$$

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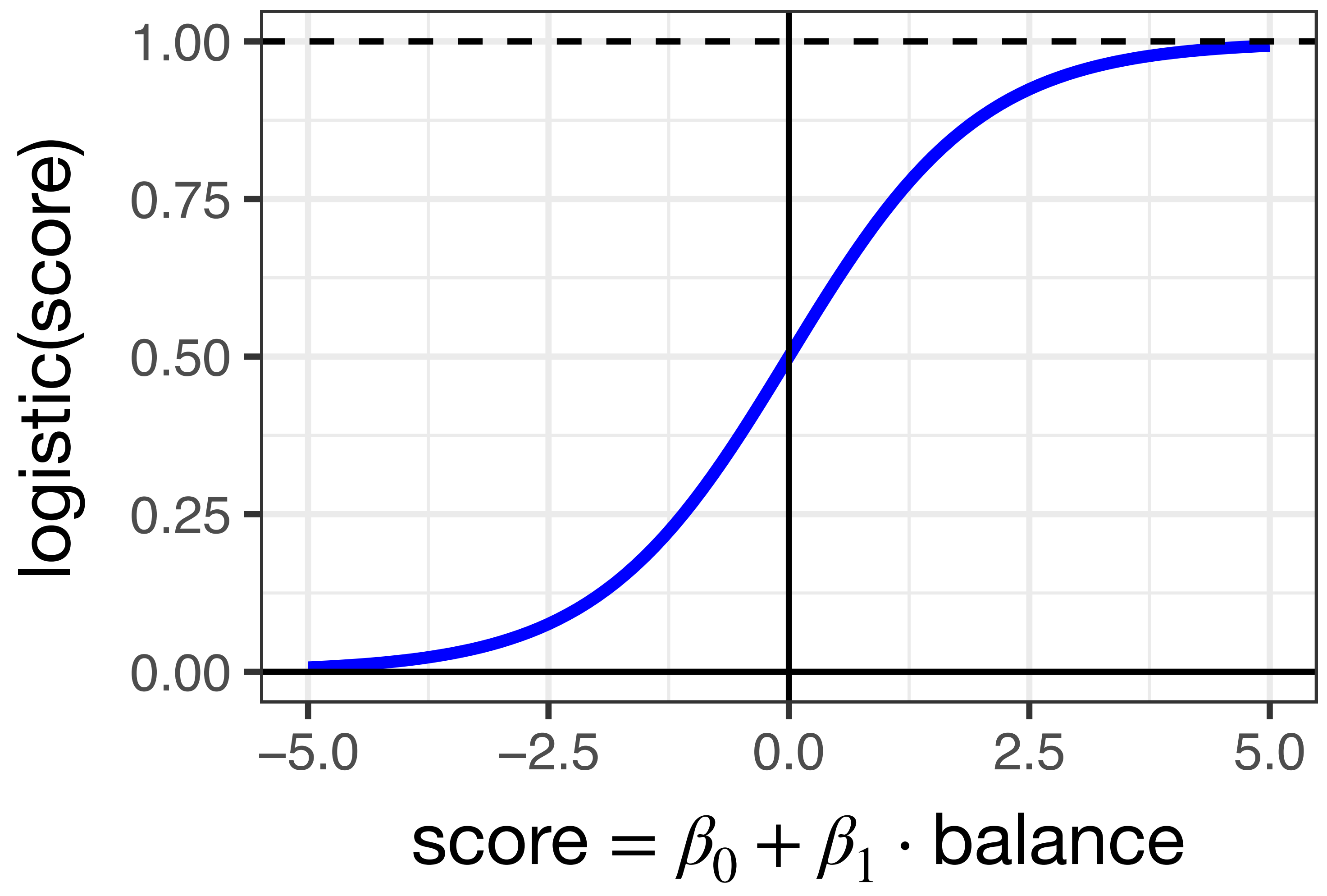

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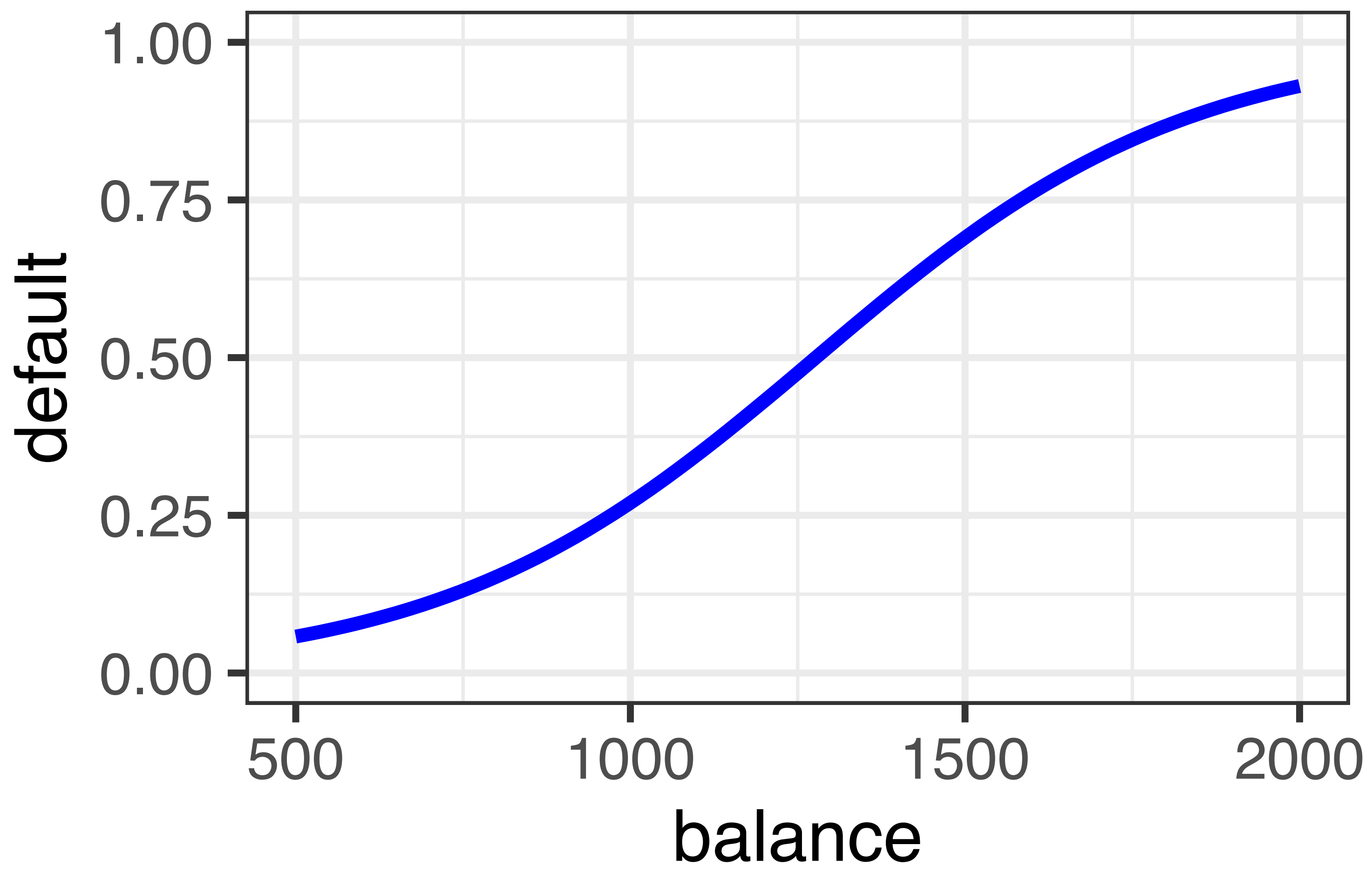
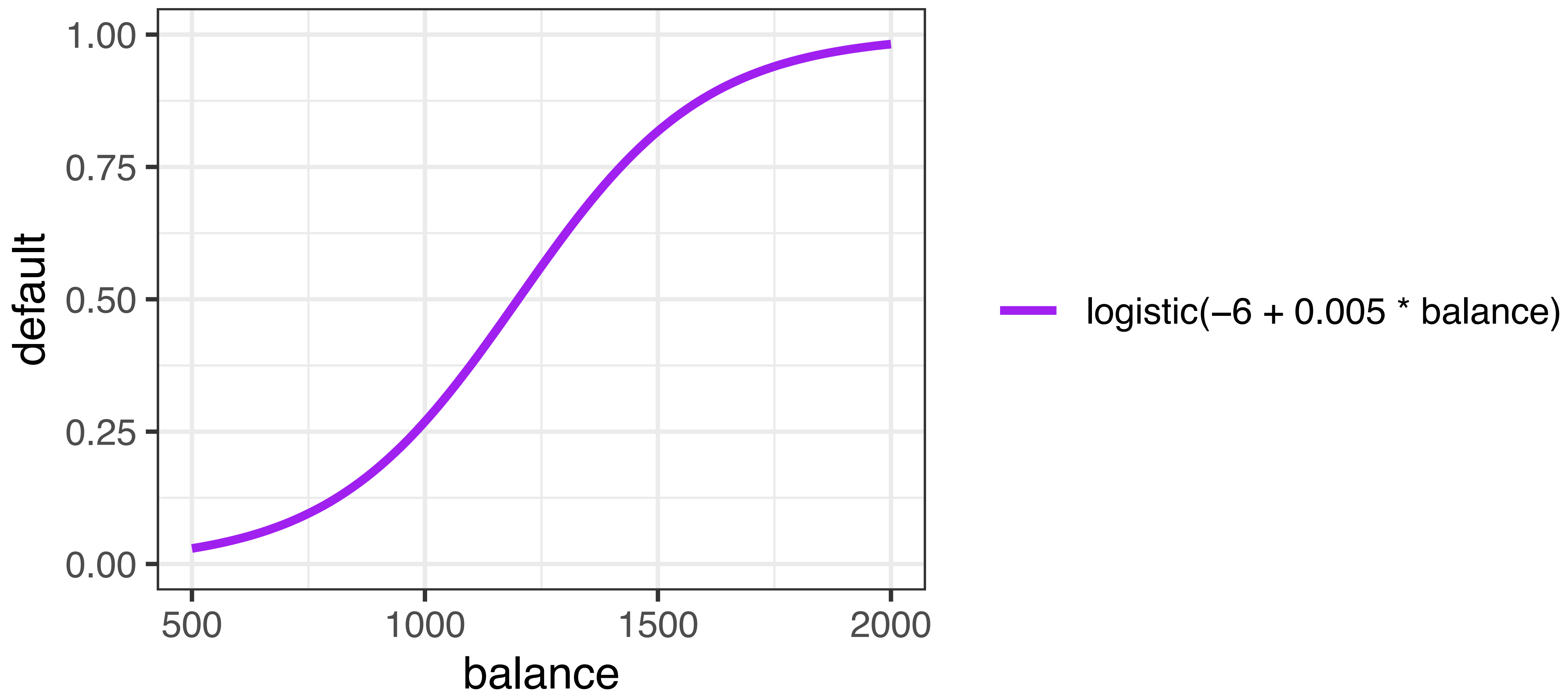


Different logistic curves



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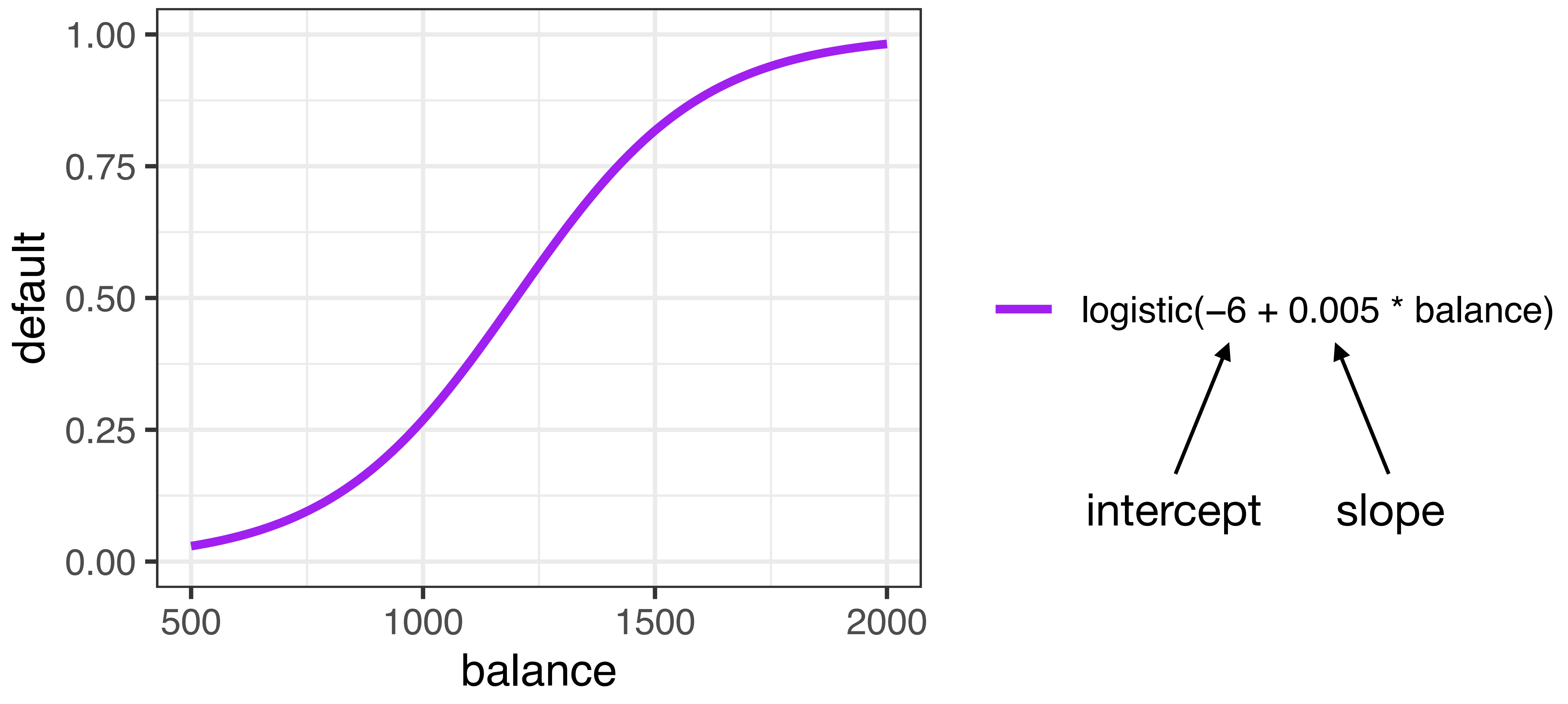

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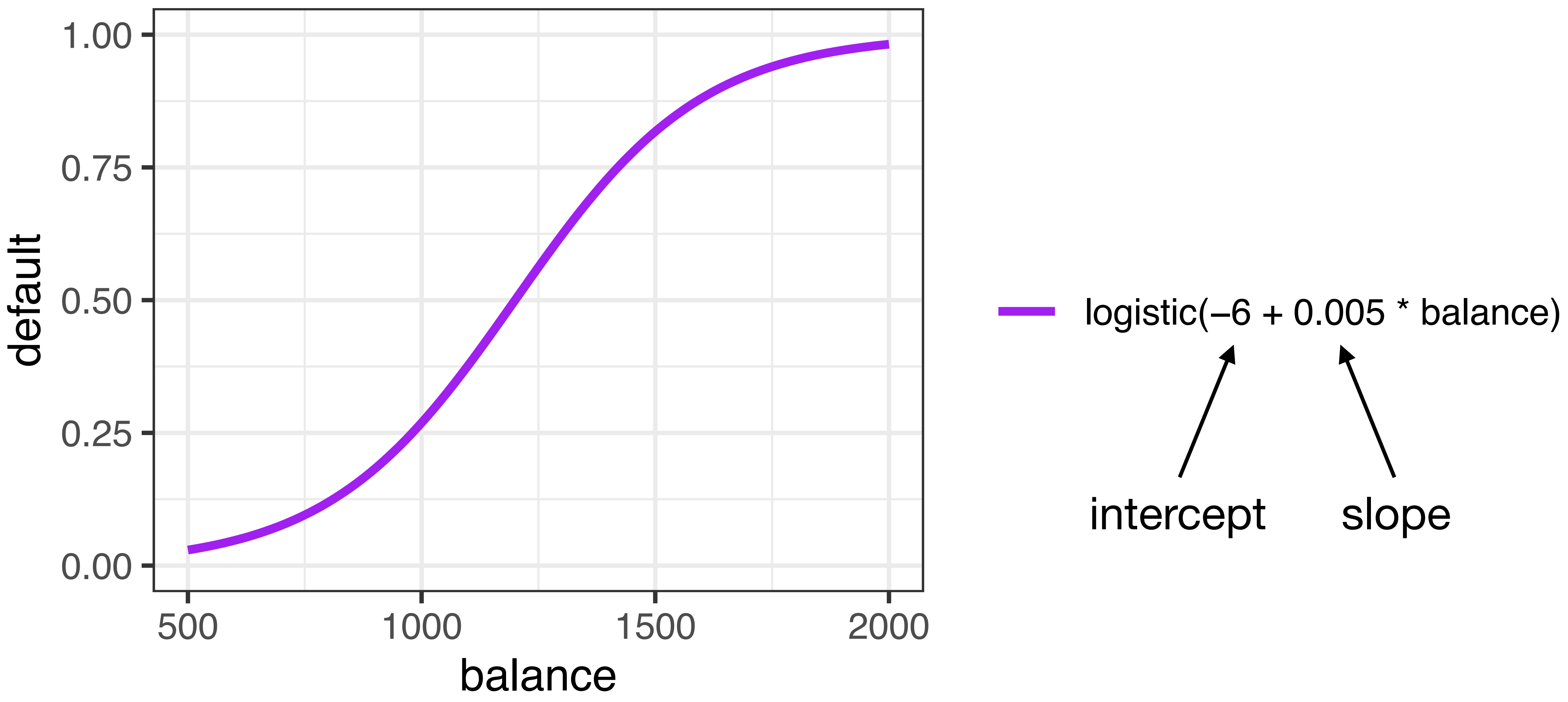

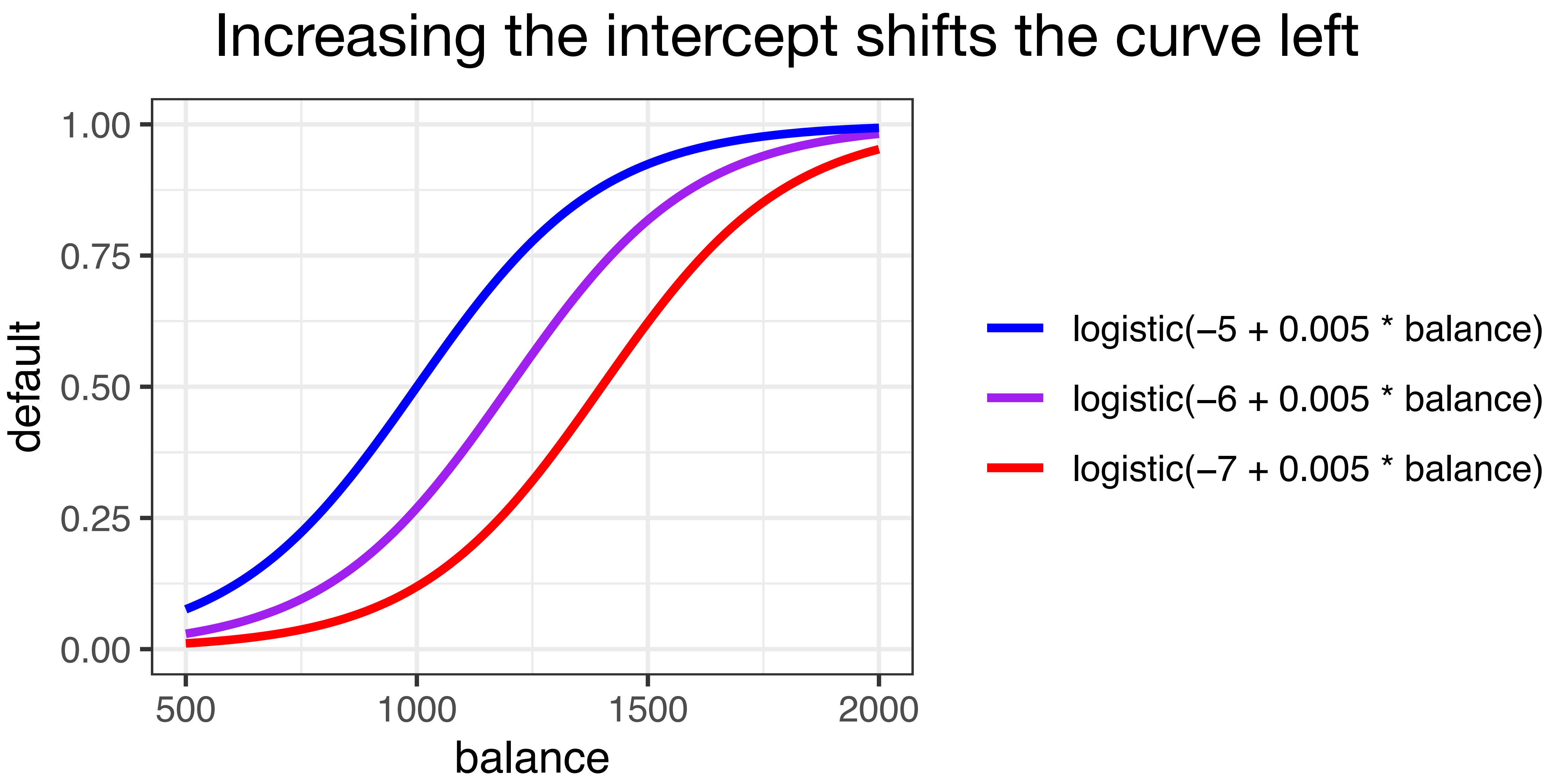

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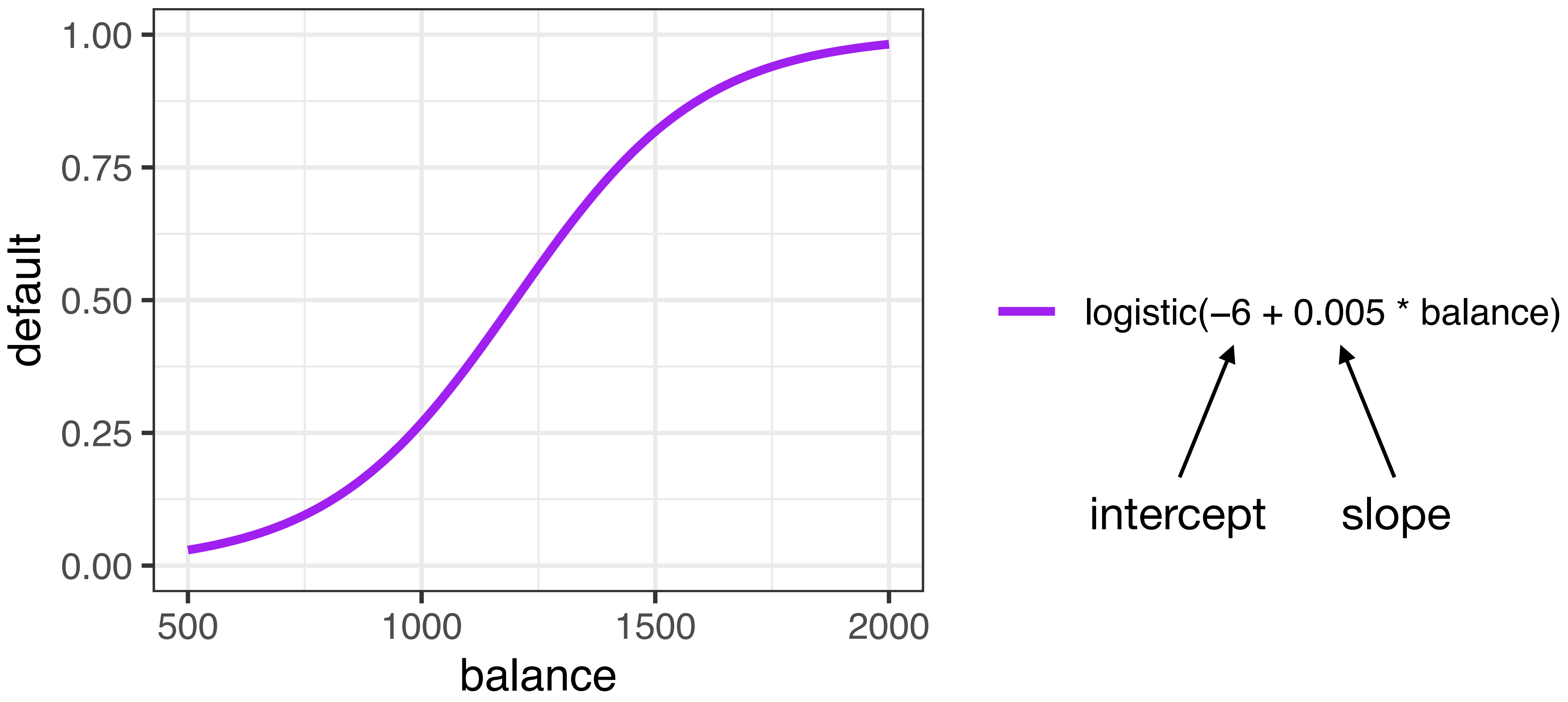
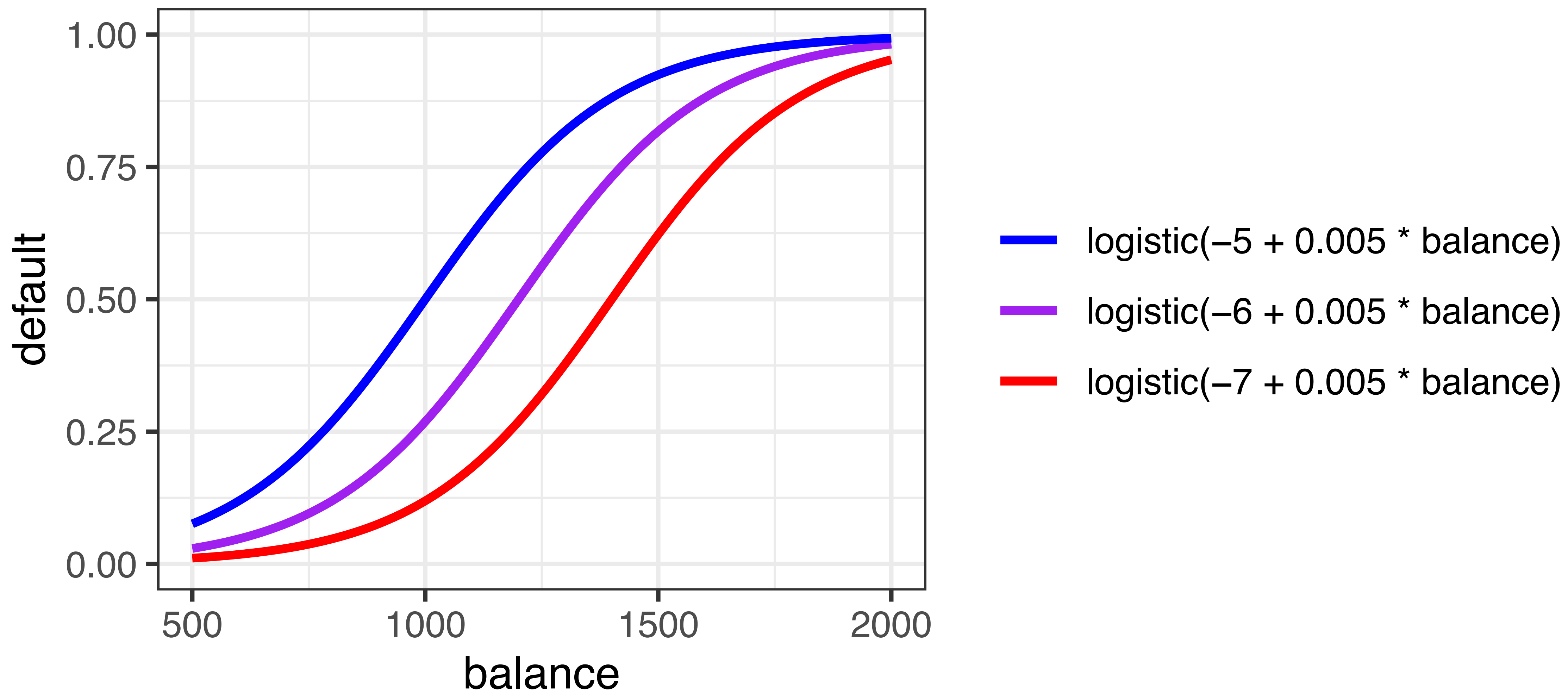
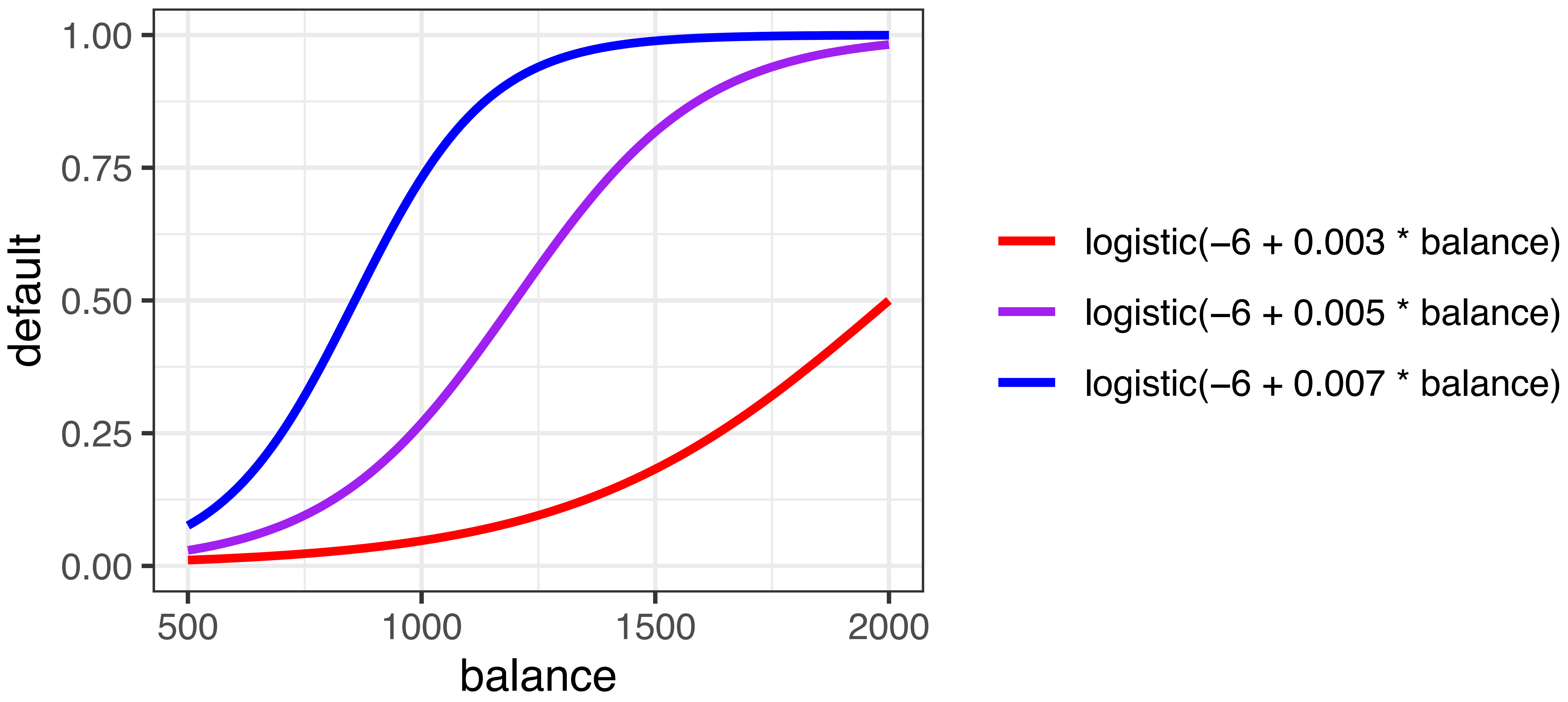


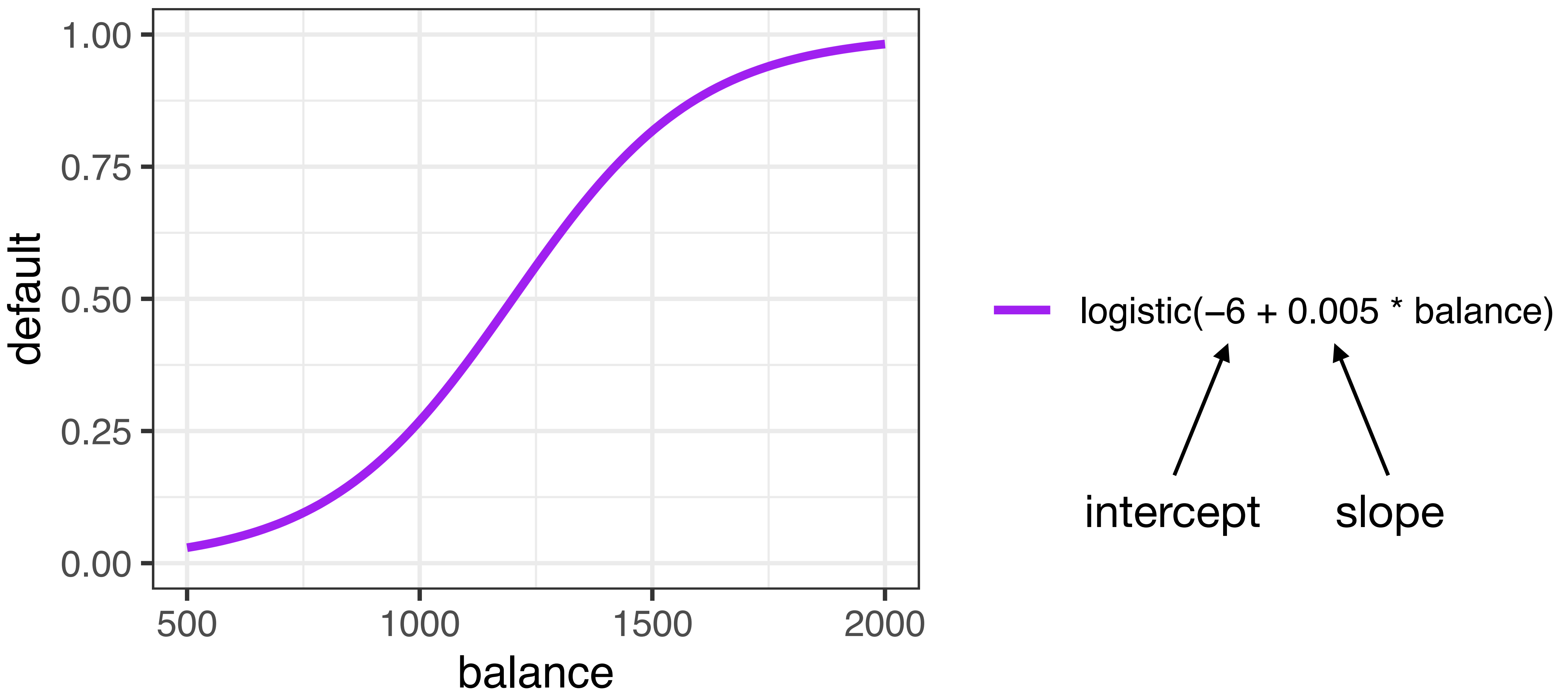
Increasing the intercept shifts the curve left

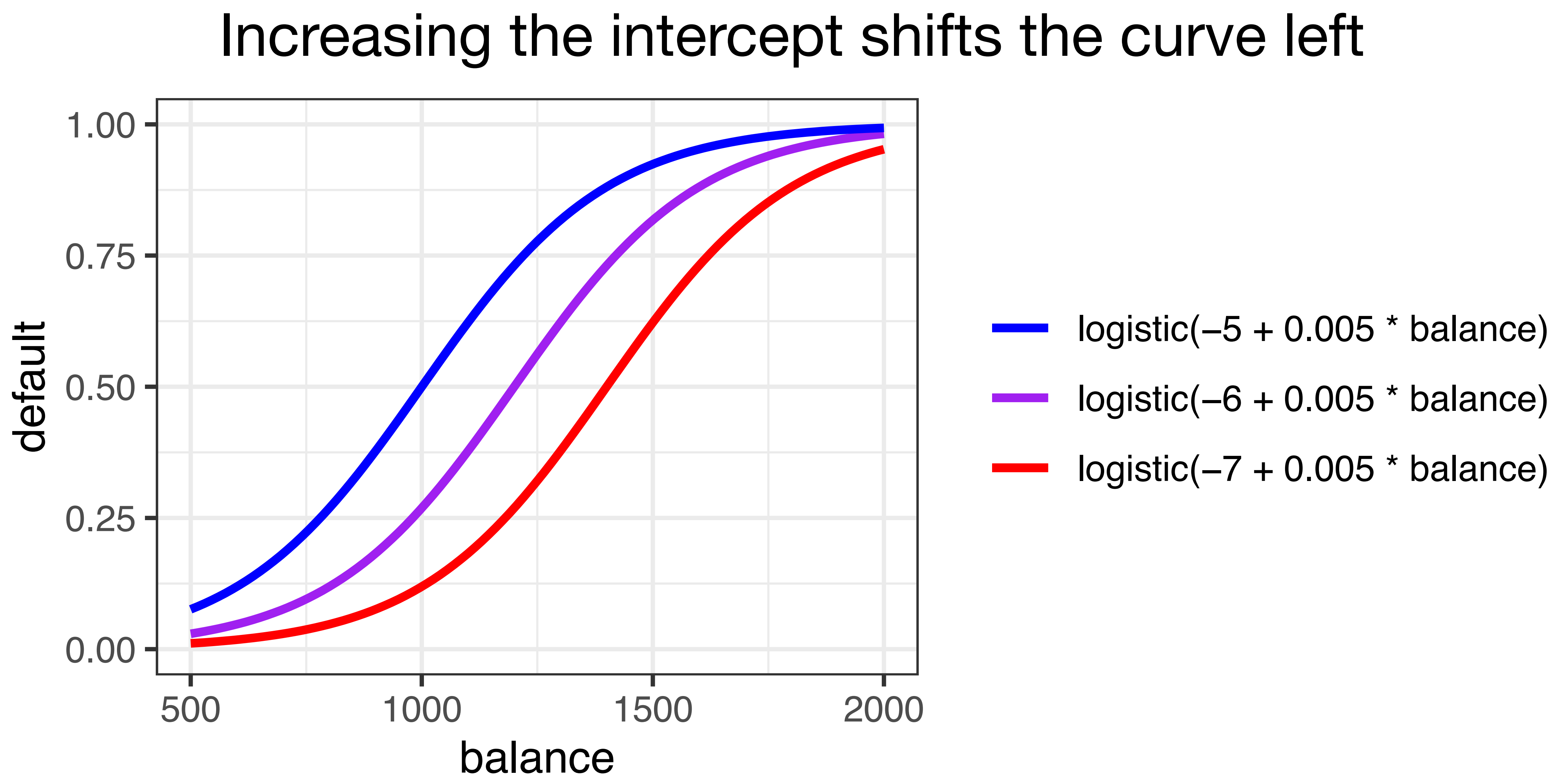


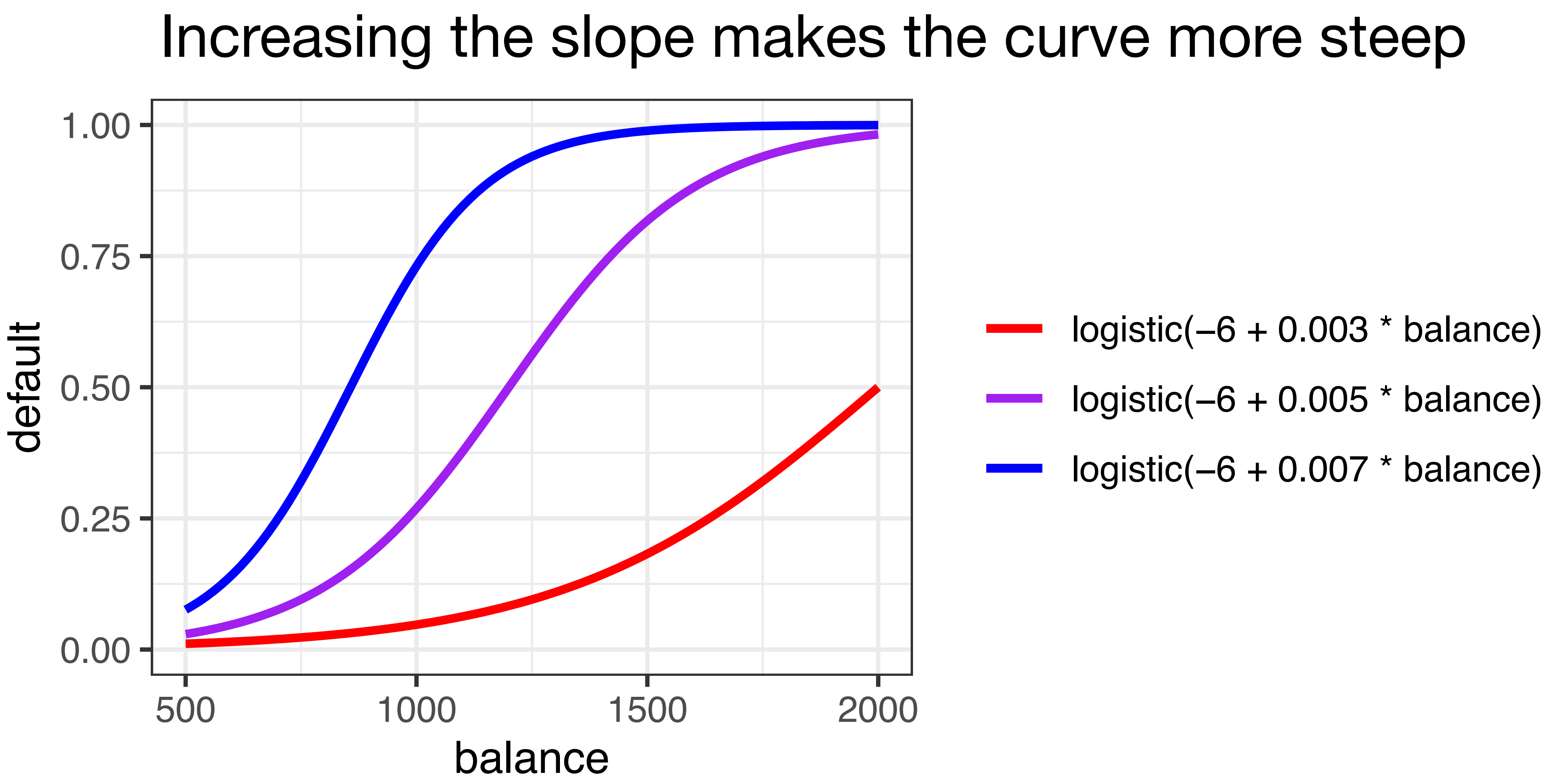
Increasing the slope makes the curve more steep

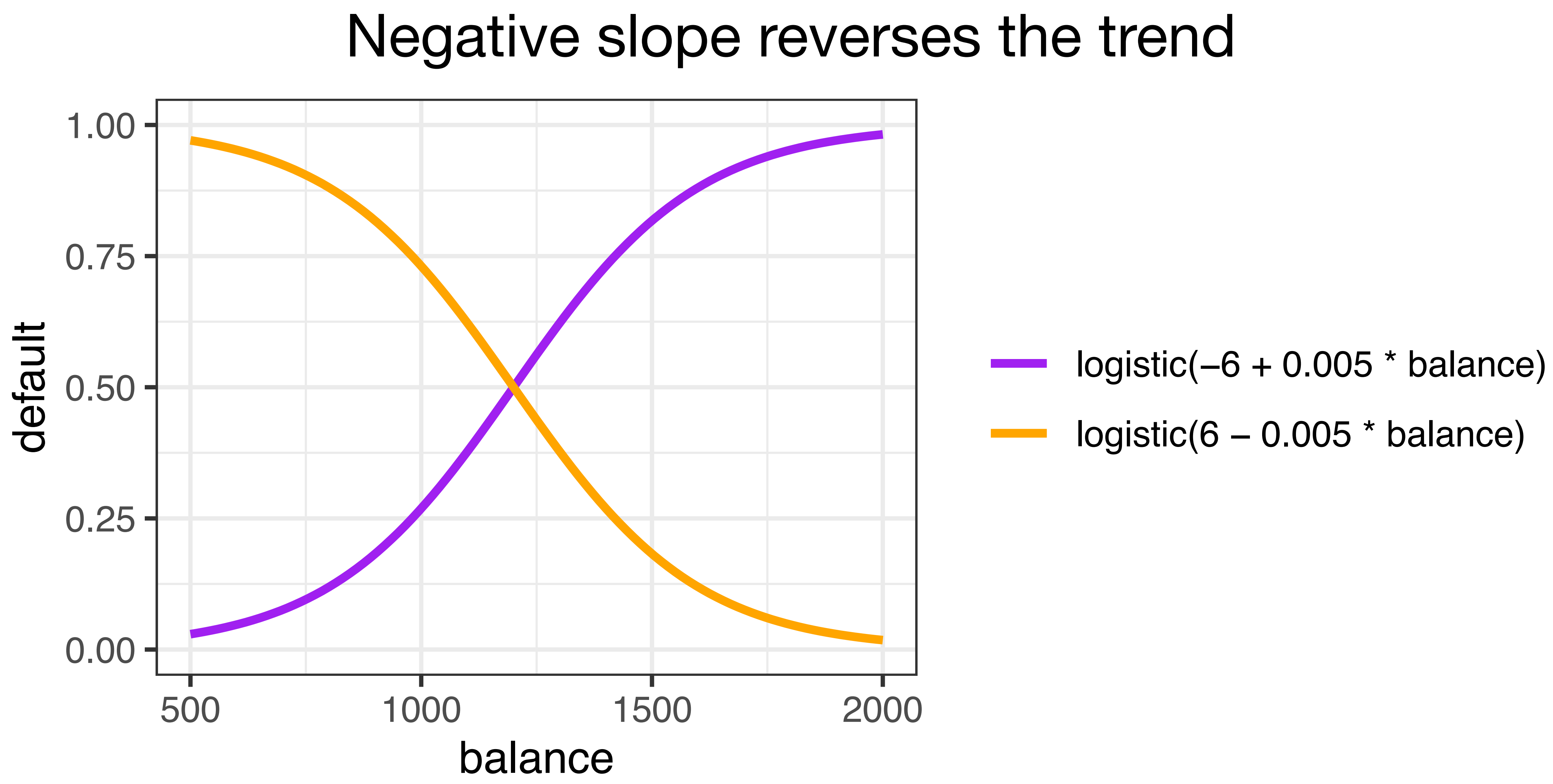


Different logistic curves

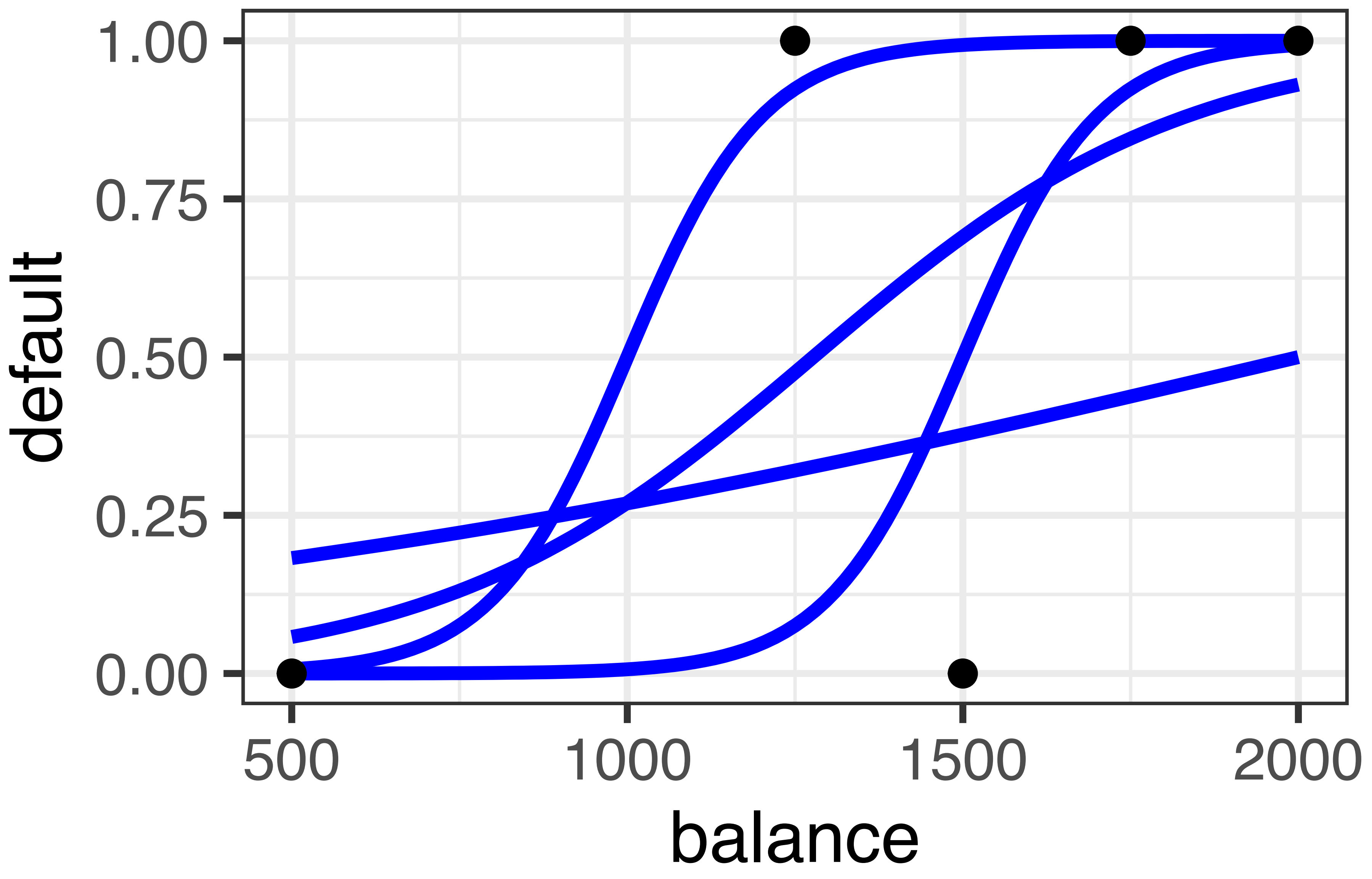








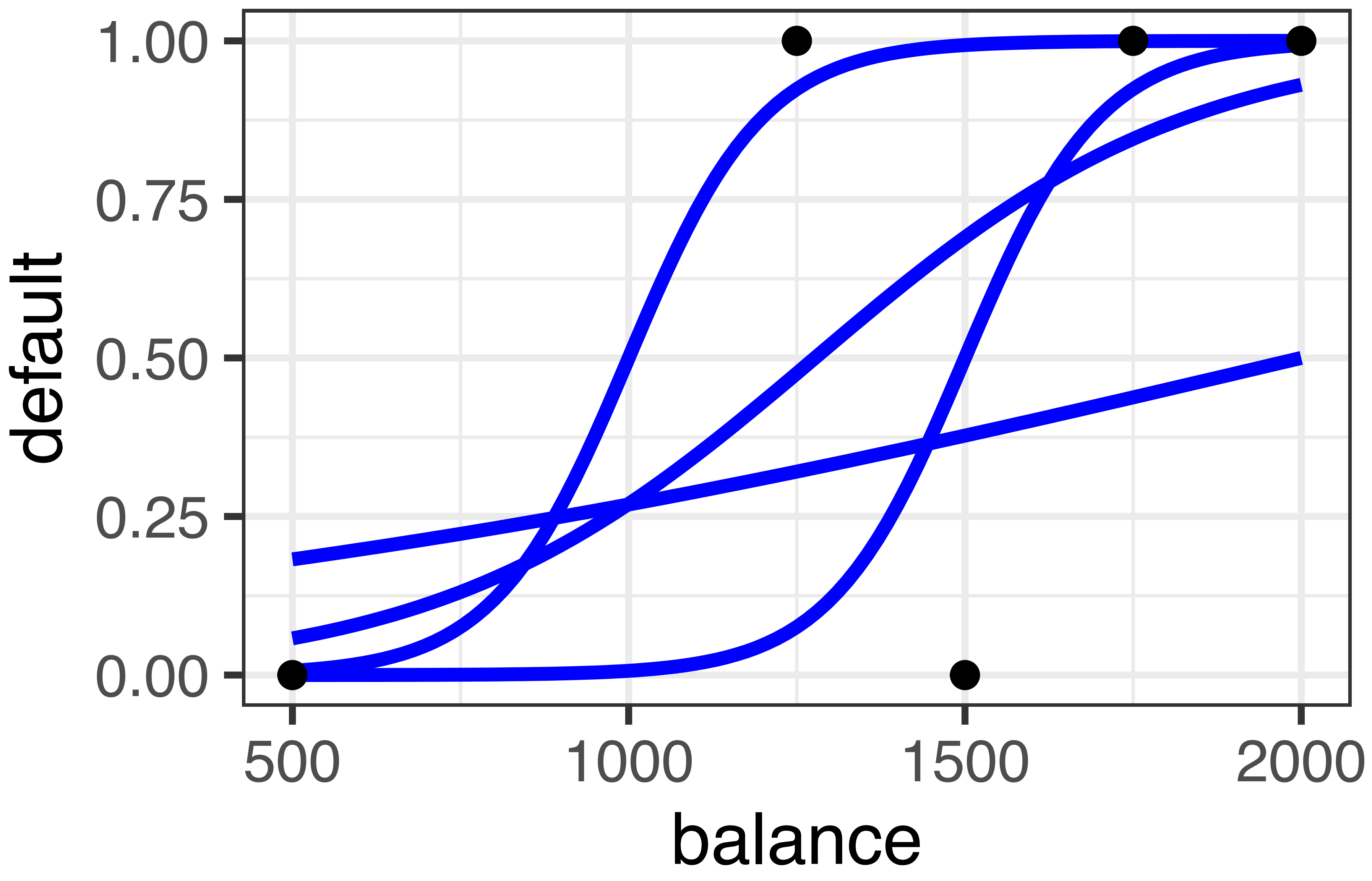
Fitting logistic regression models to data



Fitting logistic regression models to data

Each choice of (β_0, β_1) traces out a different logistic regression curve fit

$$\mathbb{P}[\text{default} \mid \text{balance}] = \text{logistic}(\beta_0 + \beta_1 \cdot \text{balance}).$$



Fitting logistic regression models to data

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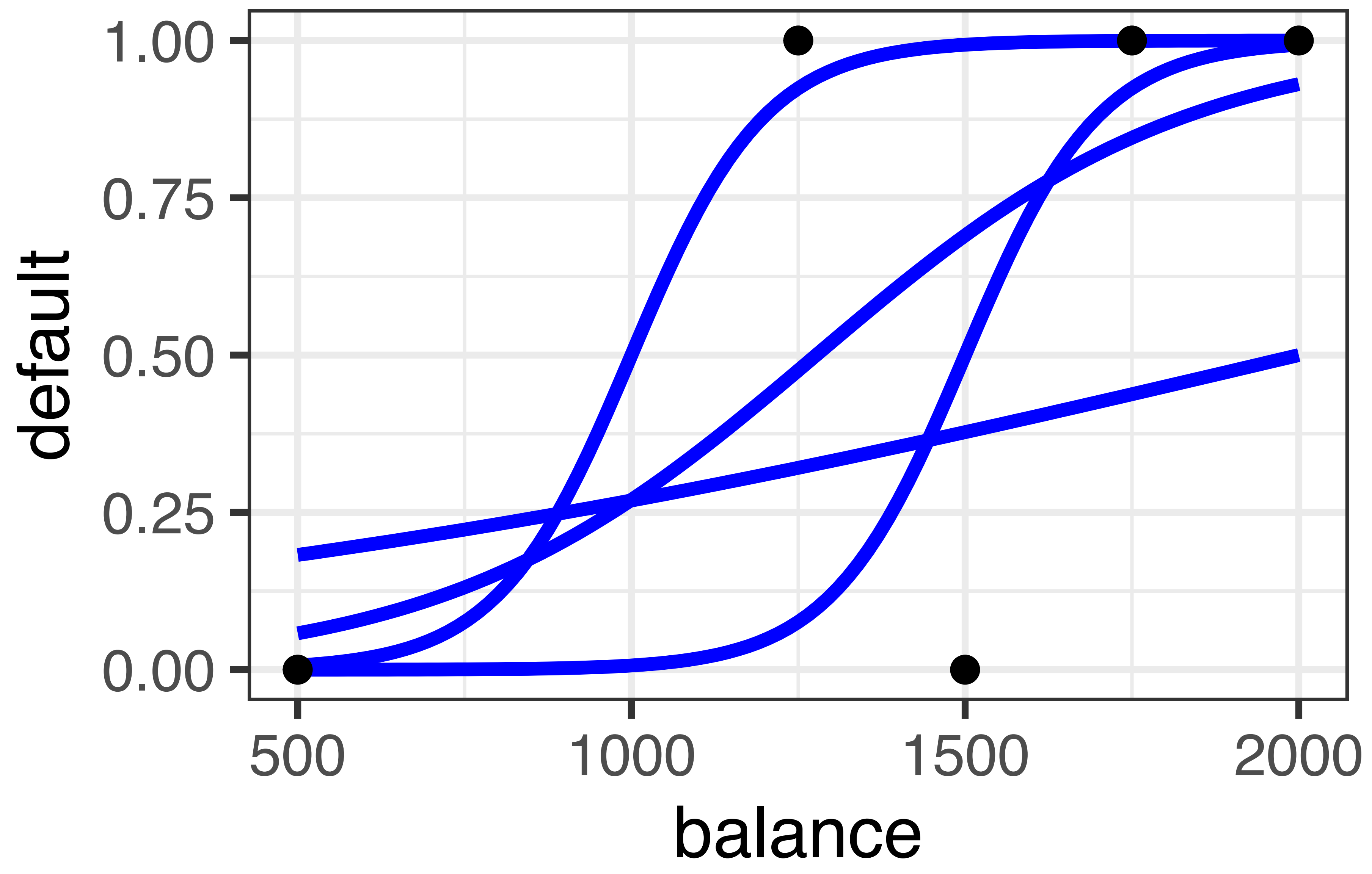
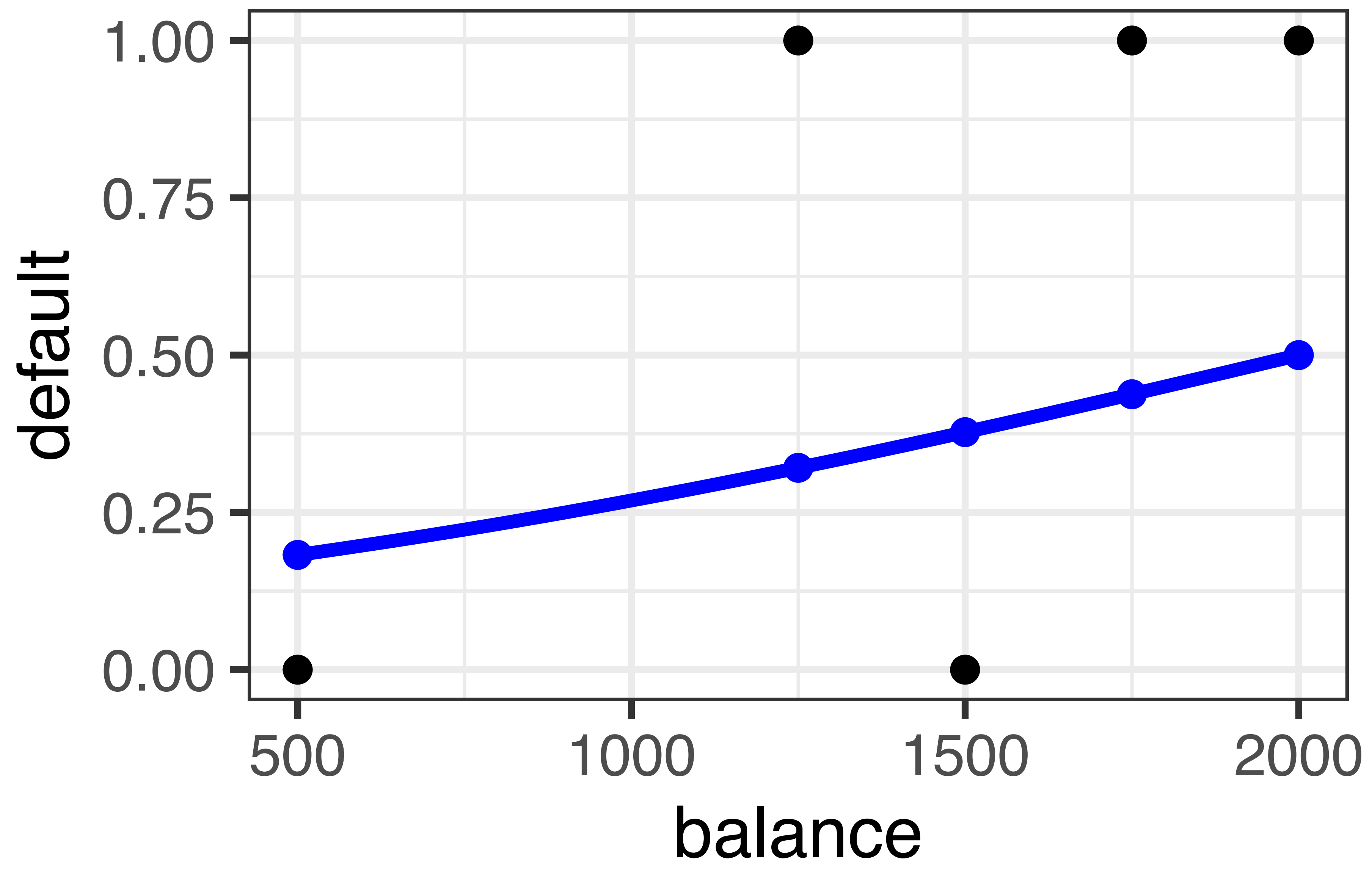
$$\mathbb{P}[\text{default} \mid \text{balance}] = \text{logistic}(\beta_0 + \beta_1 \cdot \text{balance}).$$



Which logistic regression curve fits the data the best?

Maximum likelihood estimation

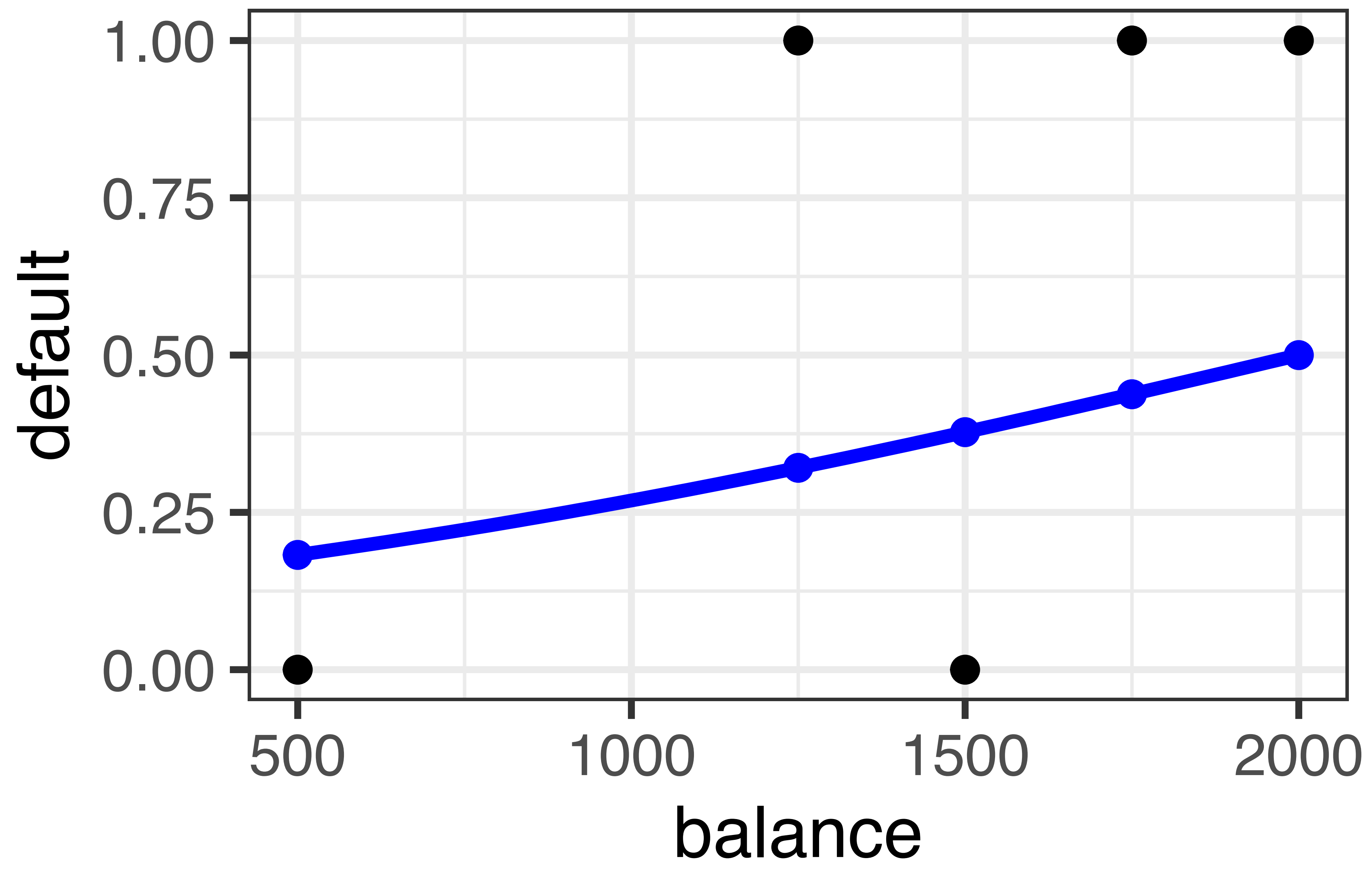
Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:



β_0	β_1
-2.0	0.001

Maximum likelihood estimation

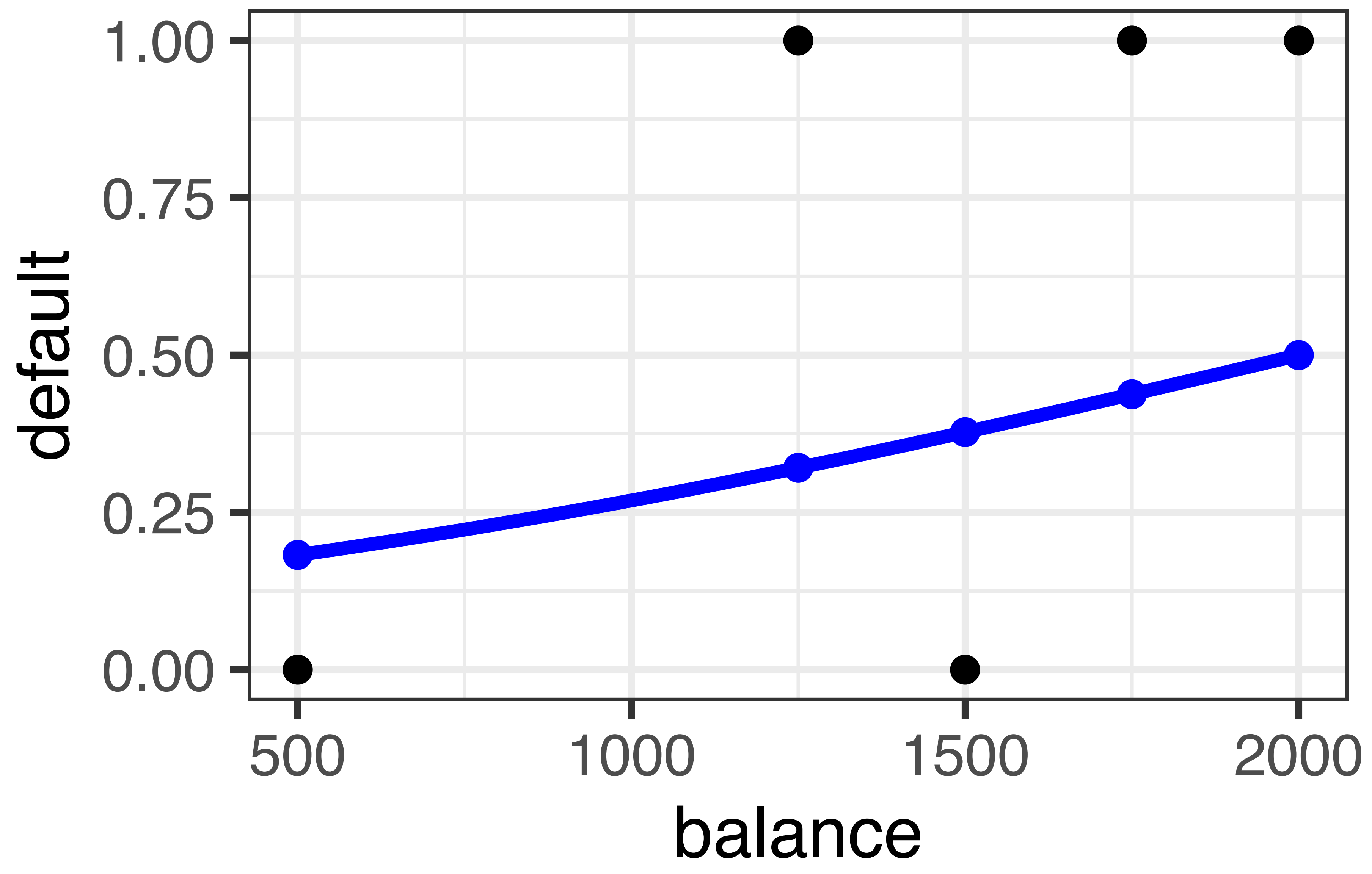
Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:



β_0	β_1	Predicted probabilities
-2.0	0.001	

Maximum likelihood estimation

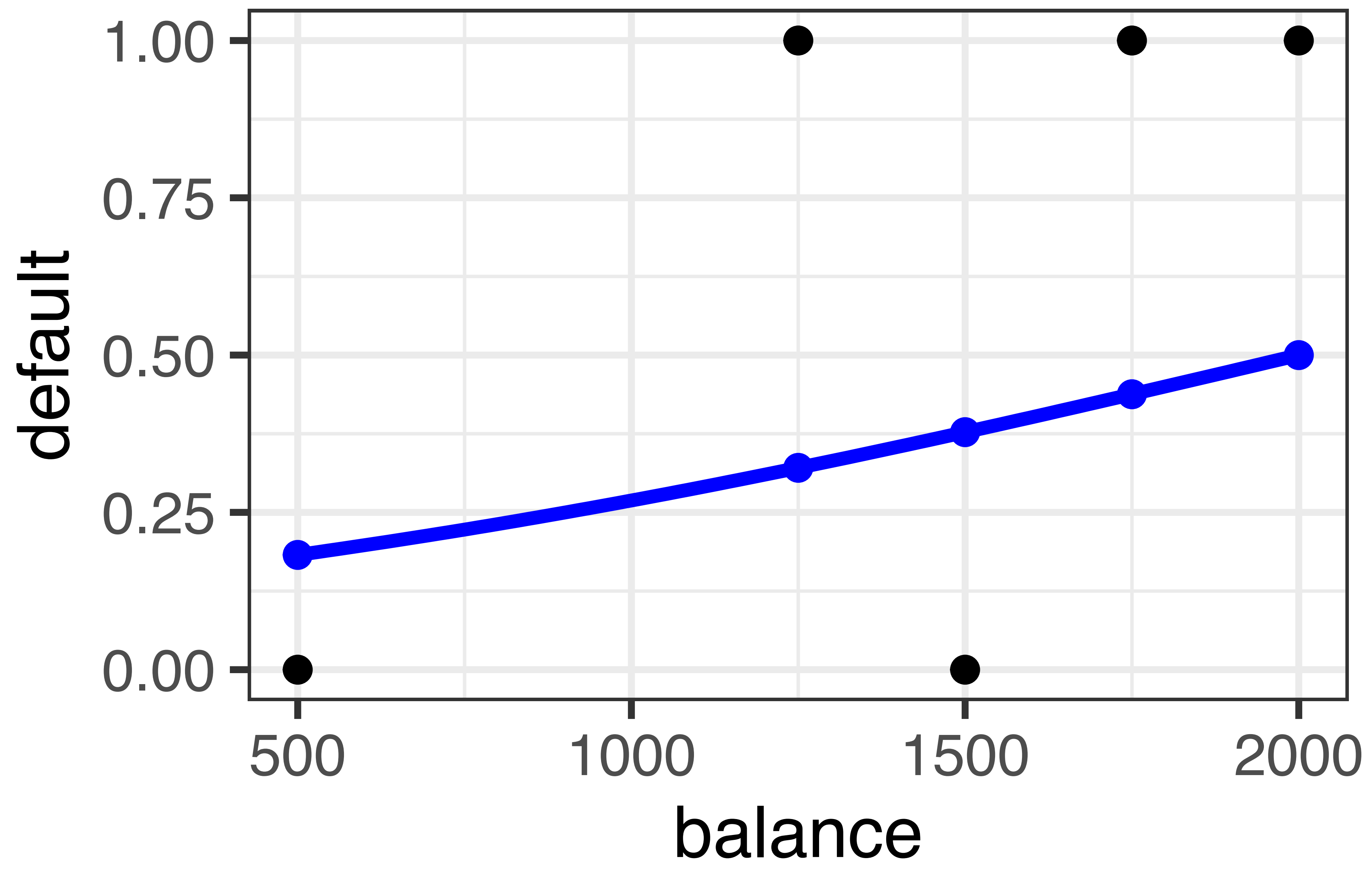
Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:



β_0	β_1	Predicted probabilities
-2.0	0.001	0.8

Maximum likelihood estimation

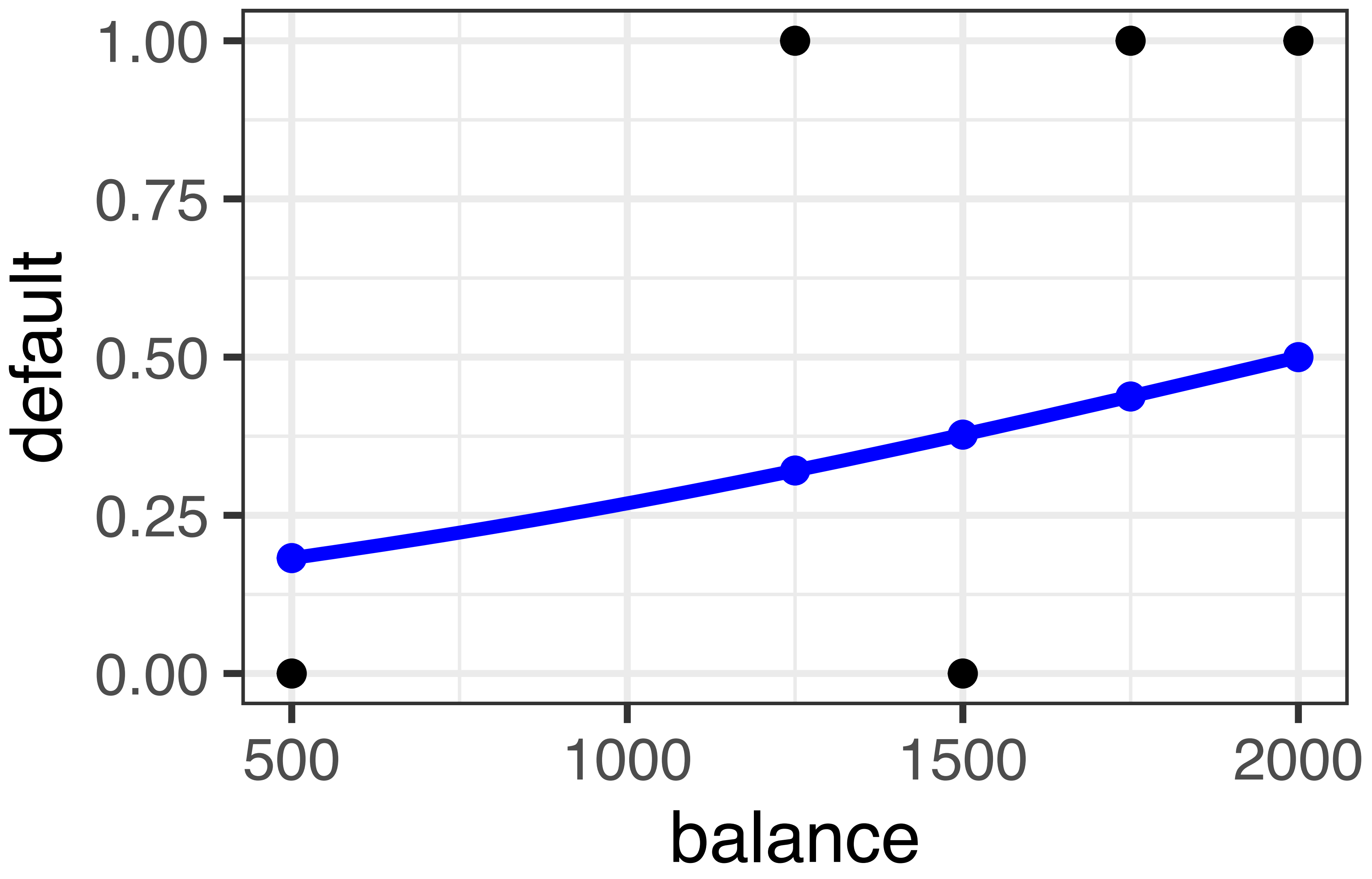
Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:



β_0	β_1	Predicted probabilities	
-2.0	0.001	0.8	0.3

Maximum likelihood estimation

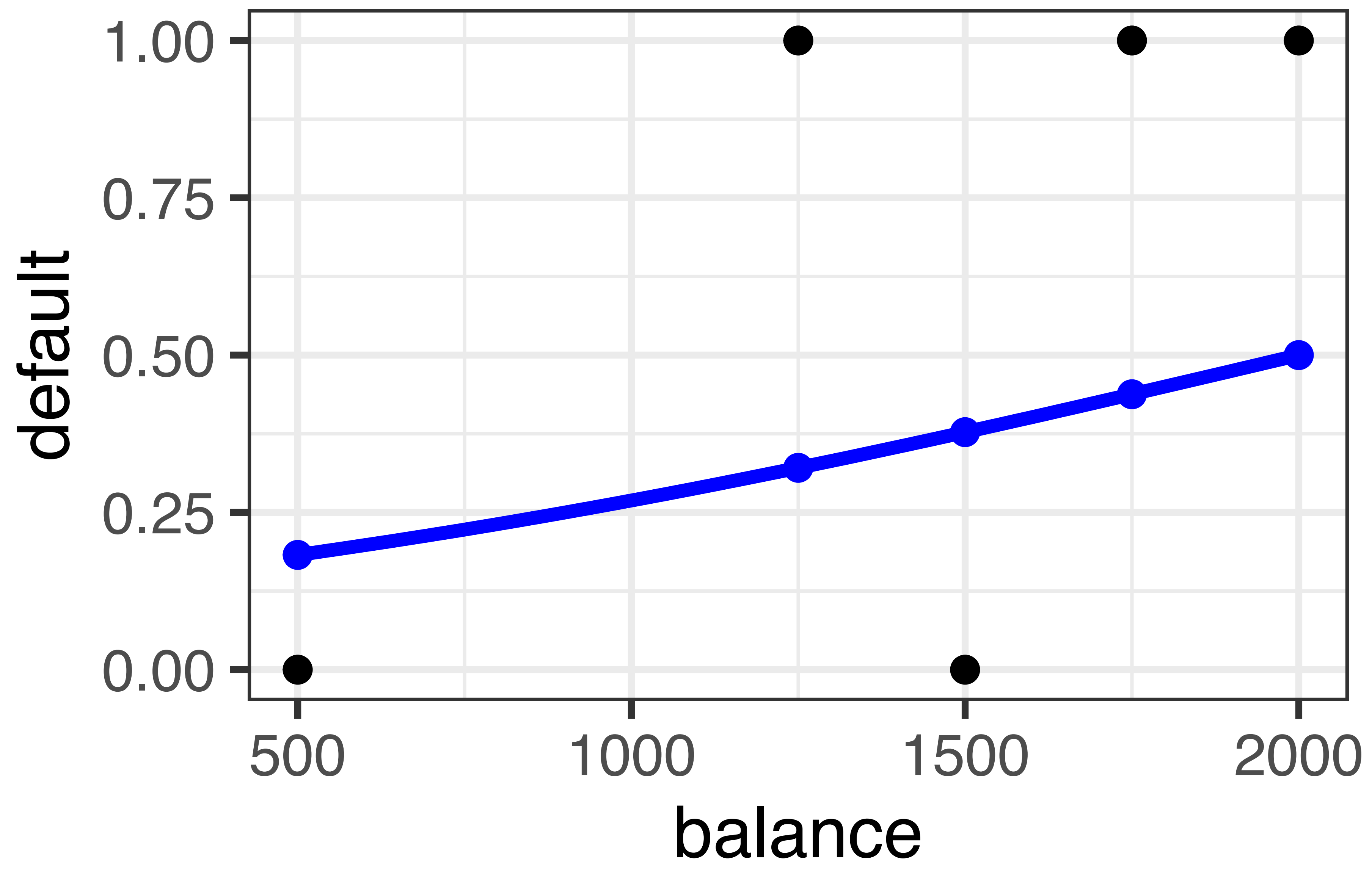
Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:



β_0	β_1	Predicted probabilities				
-2.0	0.001	0.8	0.3	0.6		

Maximum likelihood estimation

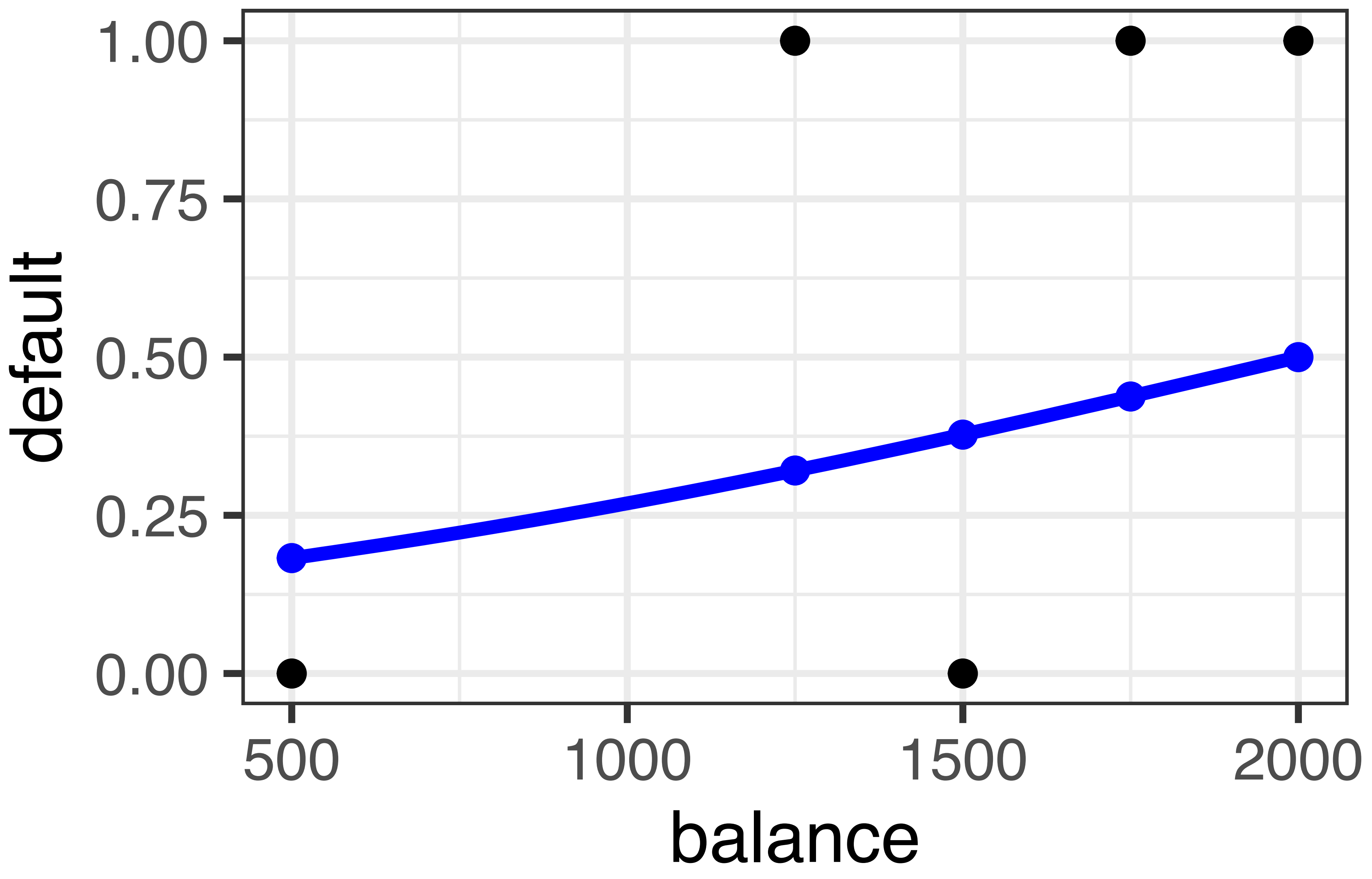
Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:



β_0	β_1	Predicted probabilities				
-2.0	0.001	0.8	0.3	0.6	0.4	0.5

Maximum likelihood estimation

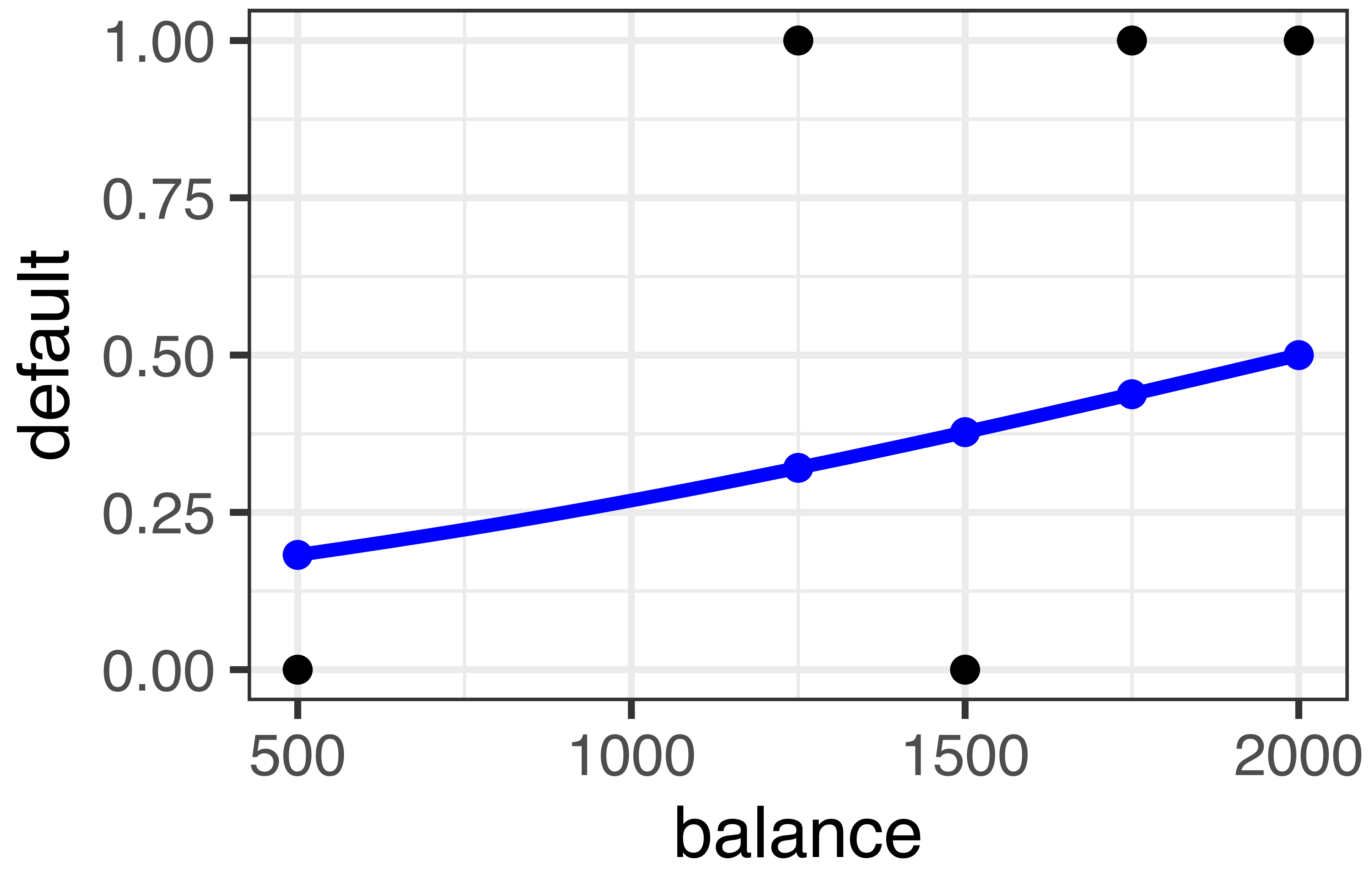
Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:



β_0	β_1	Predicted probabilities					$\mathscr{L}(\beta_0, \beta_1)$
-2.0	0.001	0.8	0.3	0.6	0.4	0.5	

Maximum likelihood estimation

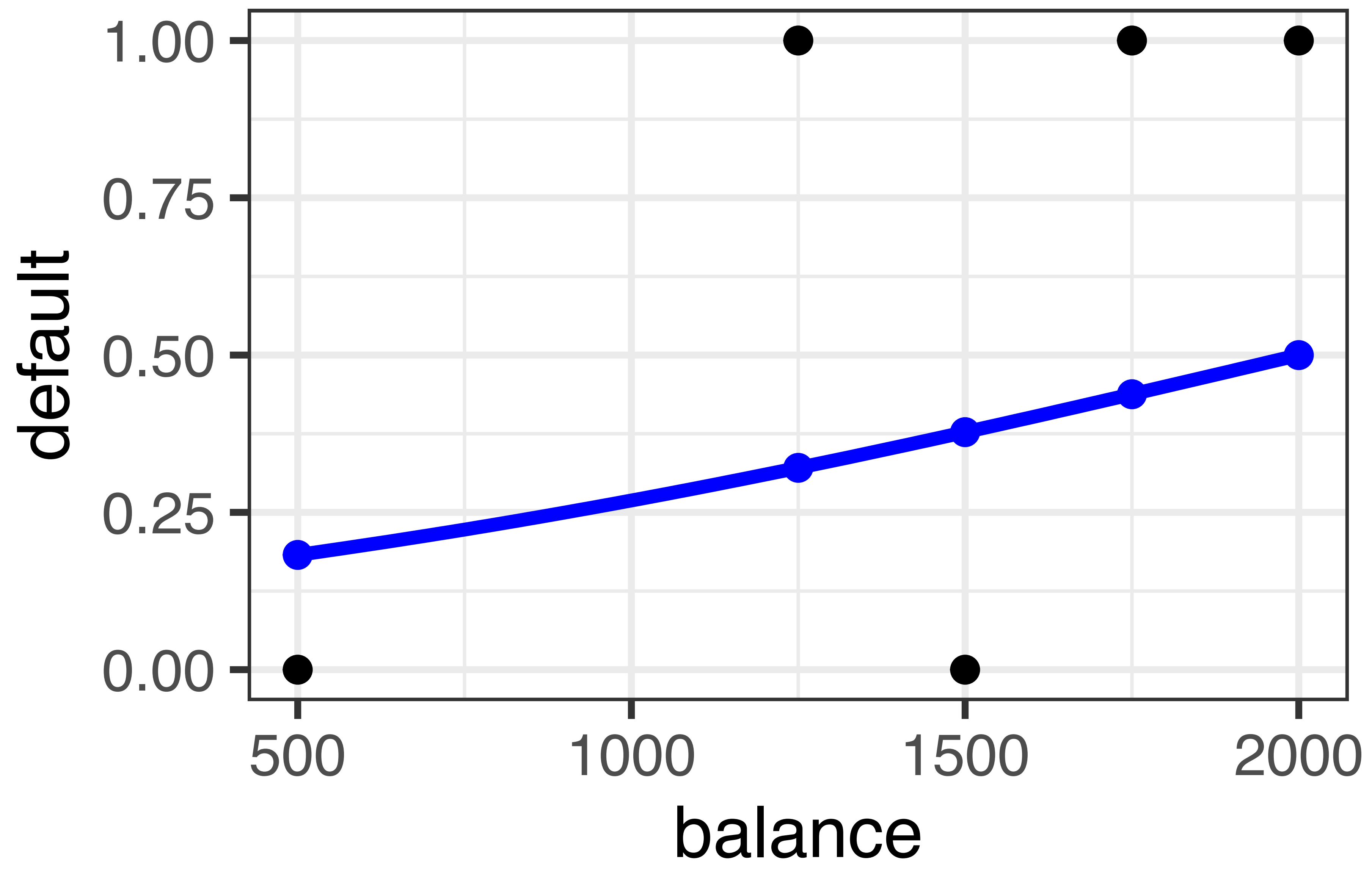
Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:



β_0	β_1	Predicted probabilities			$\mathscr{L}(\beta_0, \beta_1)$
-2.0	0.001	0.8	×	0.3×0.6×0.4×0.5	

Maximum likelihood estimation

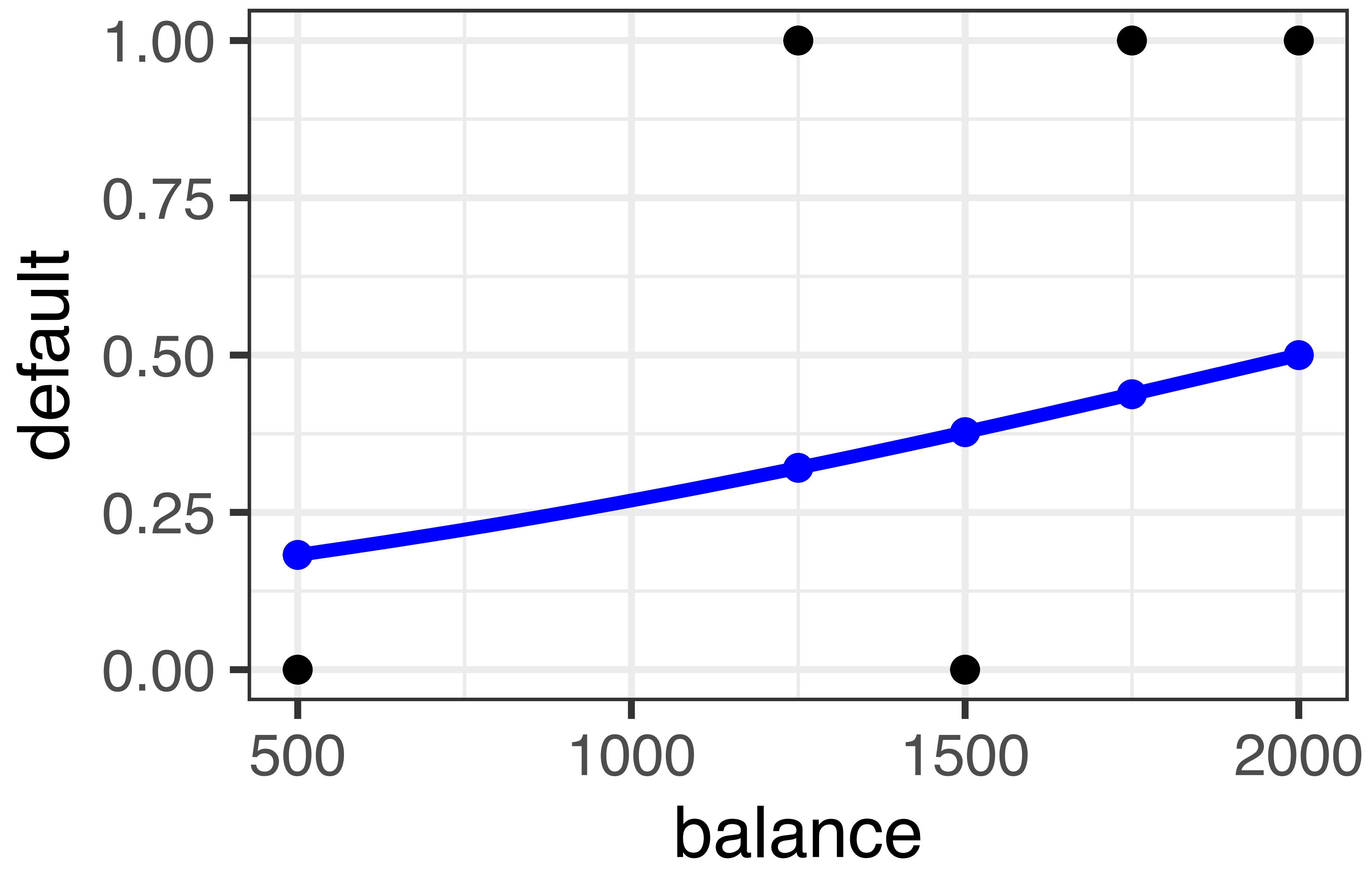
Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:



β_0	β_1	Predicted probabilities			$\mathscr{L}(\beta_0, \beta_1)$
-2.0	0.001	0.8	×	0.3×0.6×0.4×0.5	= 0.03

Maximum likelihood estimation

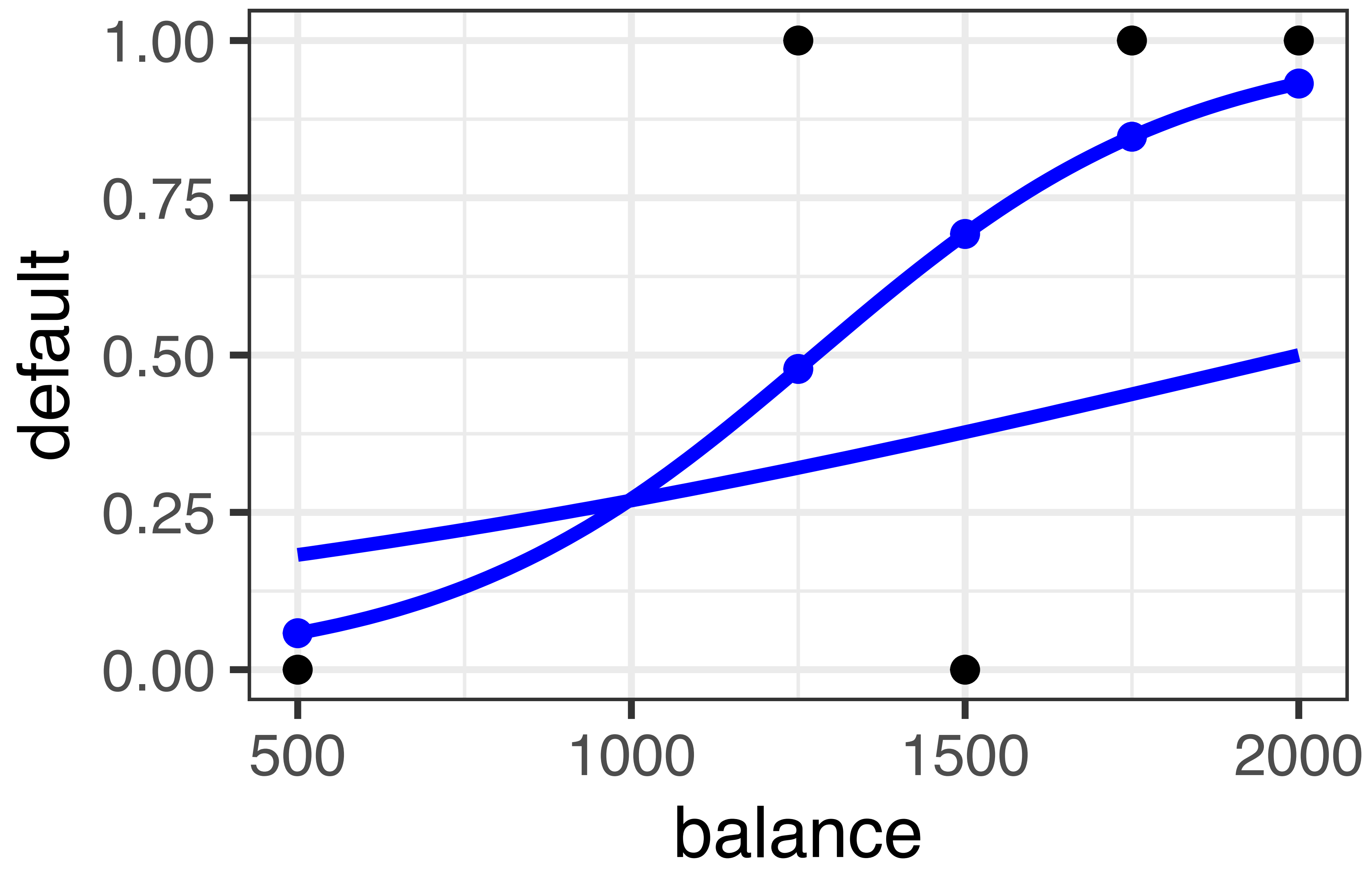
Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:



β_0	β_1	Predicted probabilities			$\mathscr{L}(\beta_0, \beta_1)$
-2.0	0.001	0.8	×	0.3×0.6×0.4×0.5	= 0.03
-4.6	0.004				

Maximum likelihood estimation

Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:



β_0	β_1	Predicted probabilities			$\mathscr{L}(\beta_0, \beta_1)$
-2.0	0.001	0.8	$\times$	0.3×0.6×0.4×0.5	= 0.03
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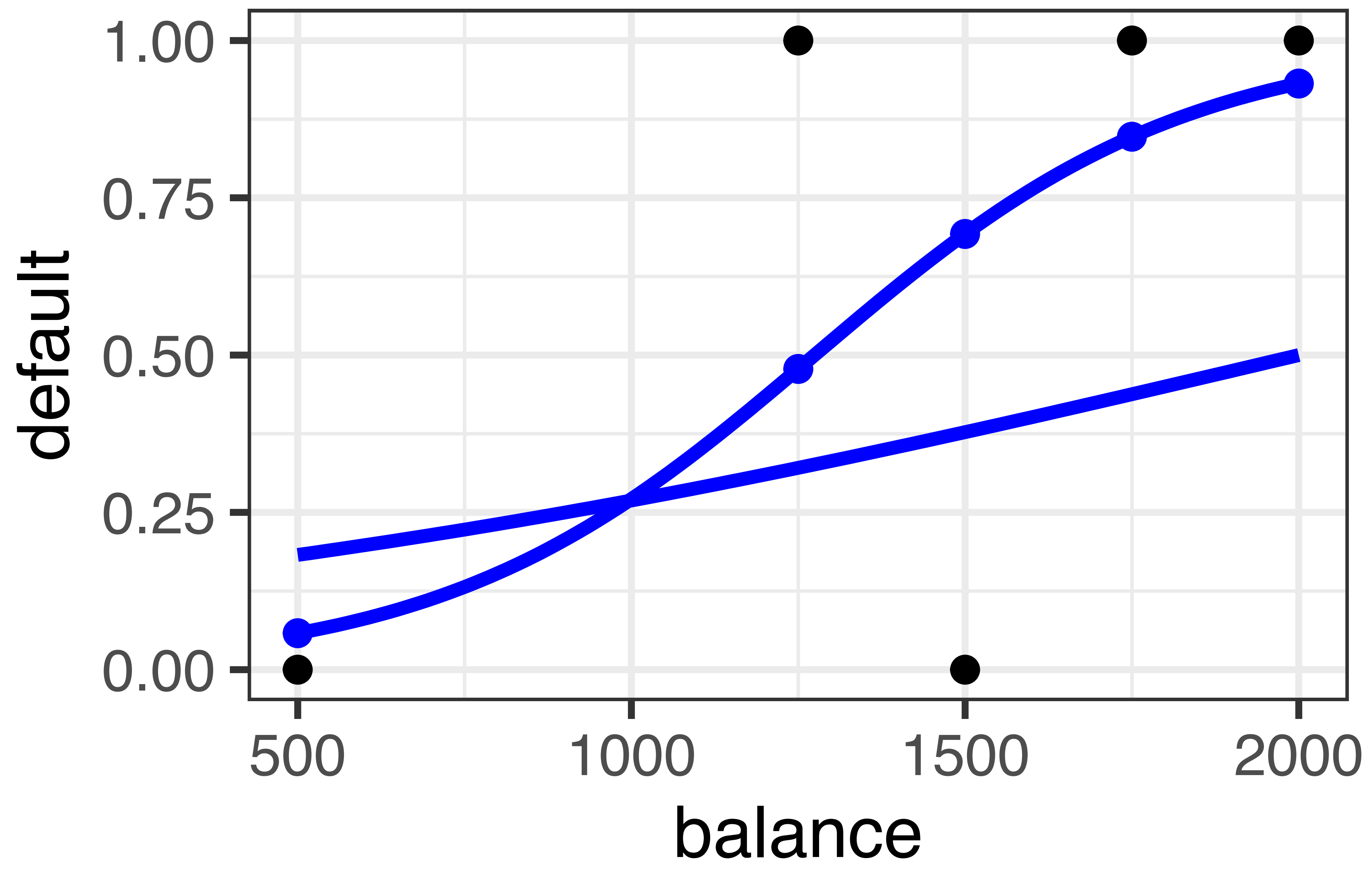
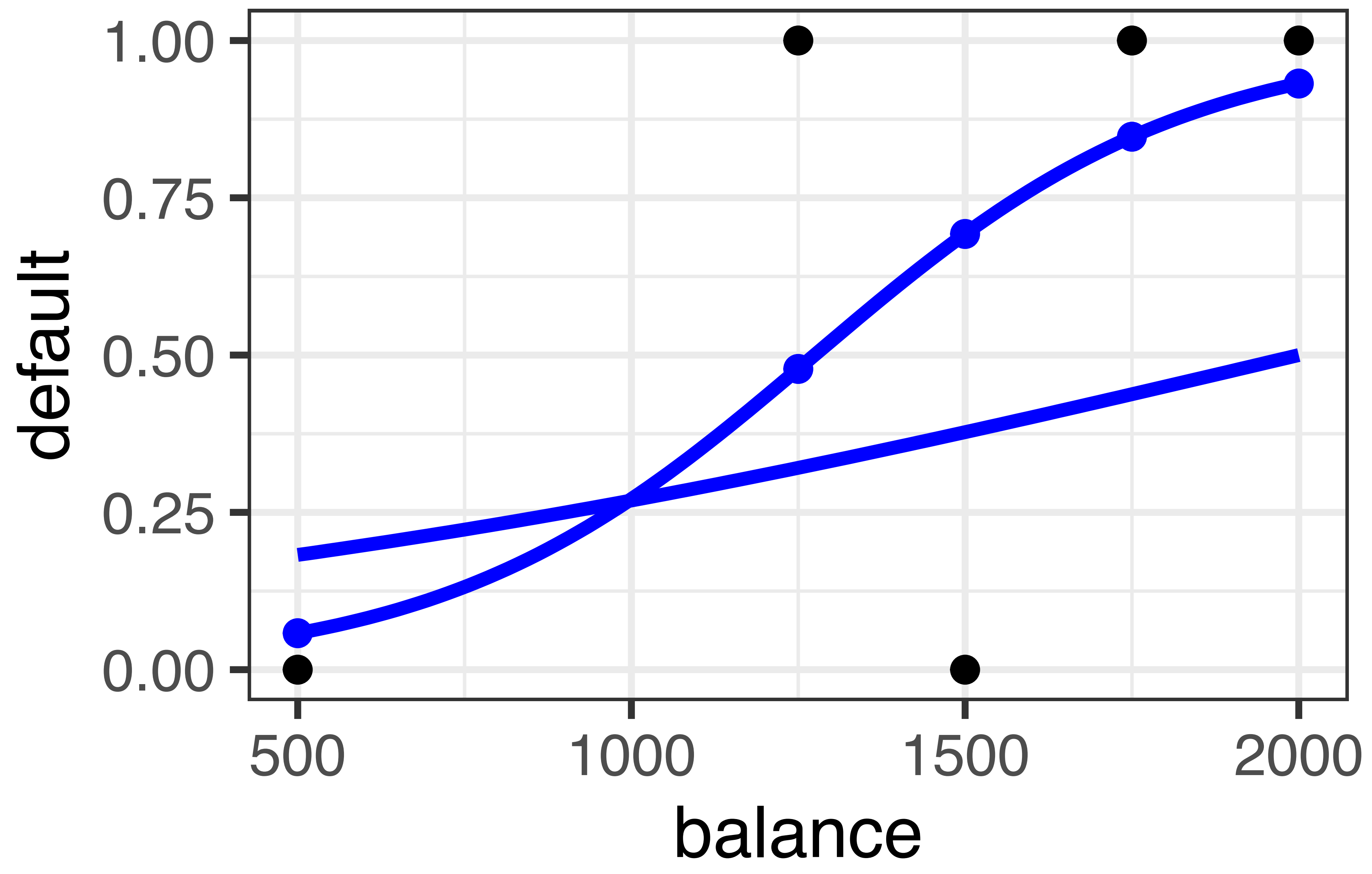


β_0	β_1	Predicted probabilities	$\mathscr{L}(\beta_0, \beta_1)$
-2.0	0.001	0.8 × 0.3×0.6×0.4×0.5	= 0.03
-4.6	0.004	0.9 0.5 0.3 0.8 0.9	

Maximum likelihood estimation

Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:



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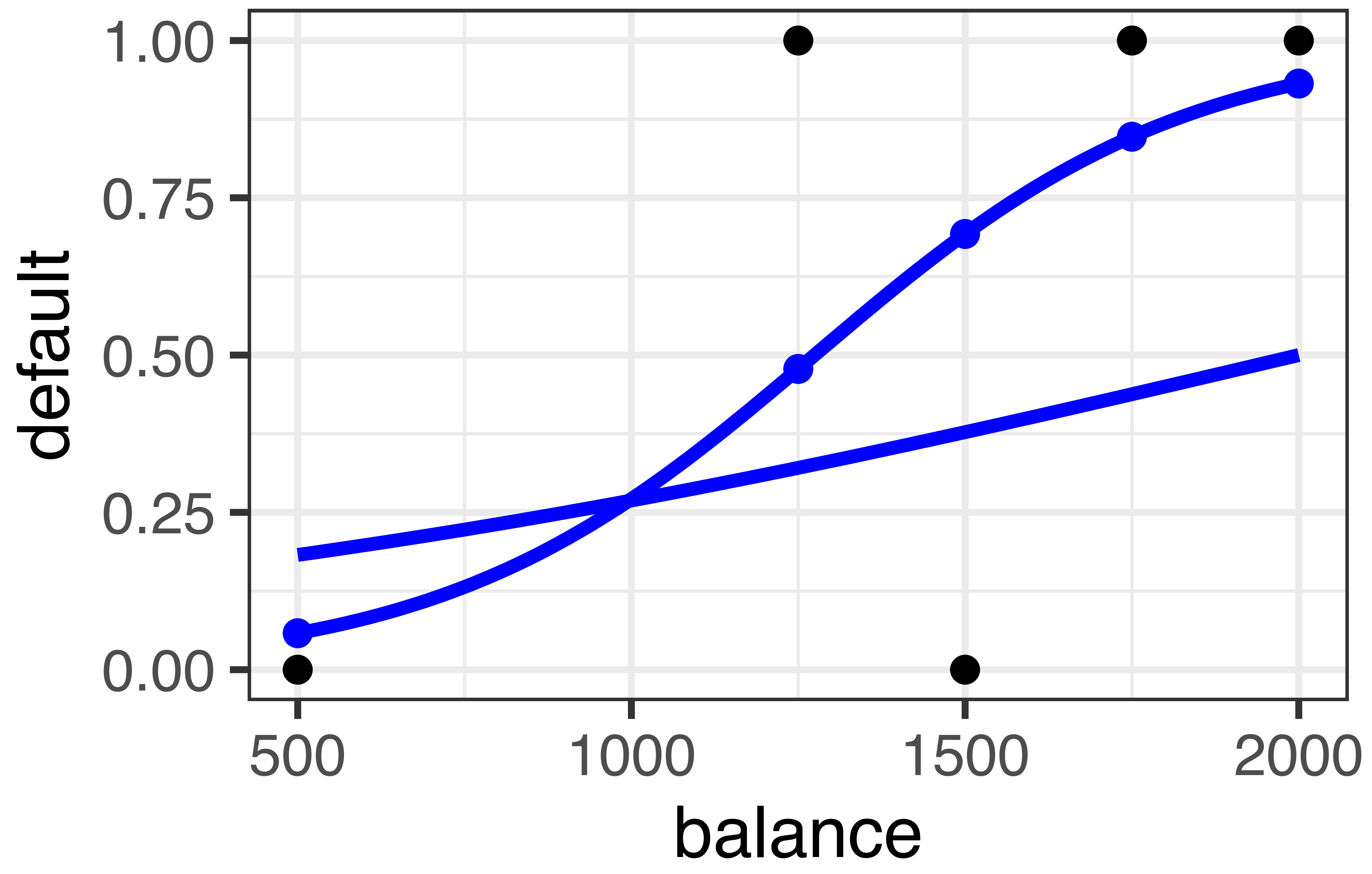

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-4.6	0.004	0.9	×	0.5×0.3×0.8×0.9	= 0.1

Maximum likelihood estimation

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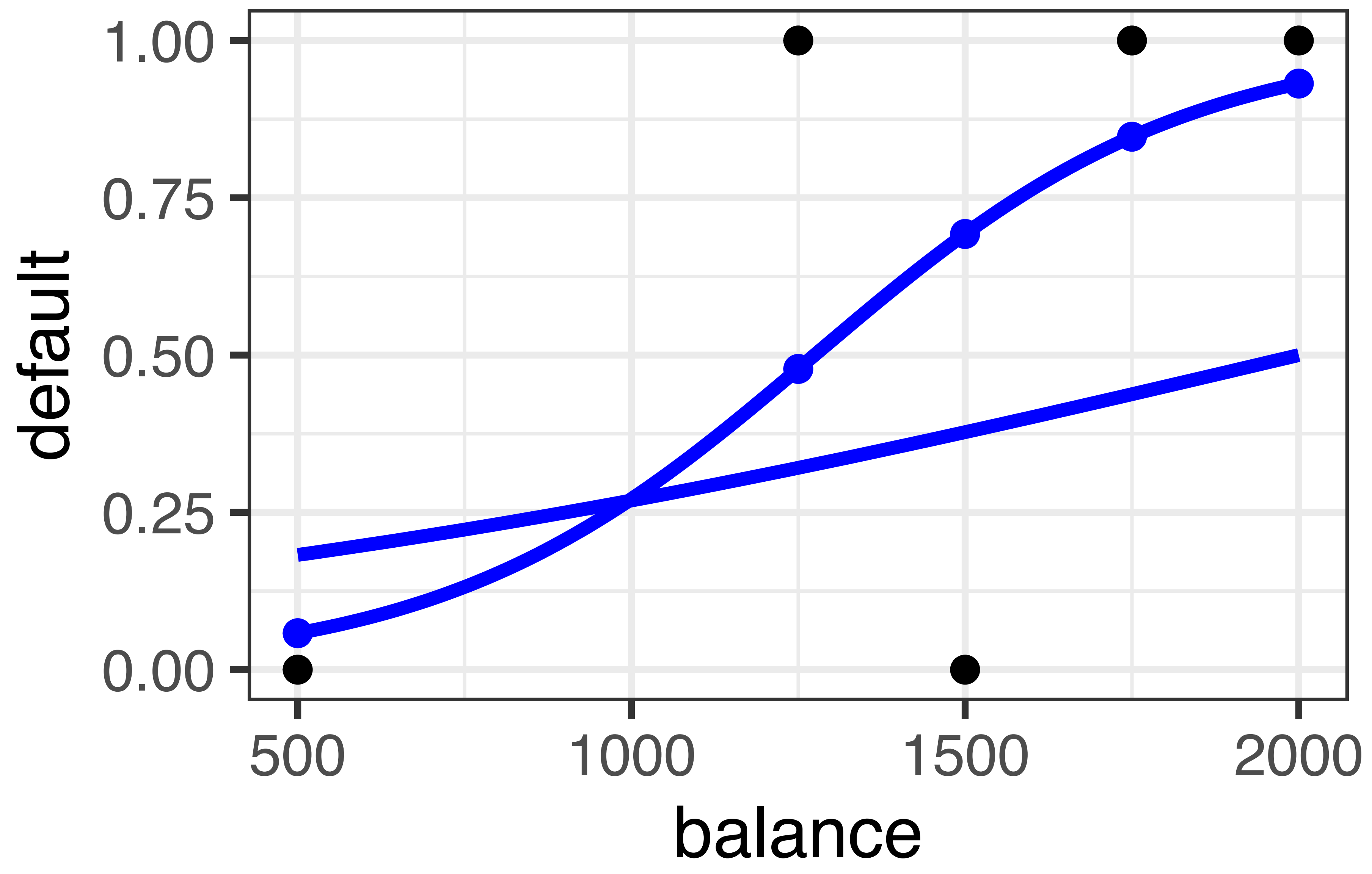

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-4.6	0.004	0.9	×	0.5×0.3×0.8×0.9	= 0.1
-15.0	0.01				

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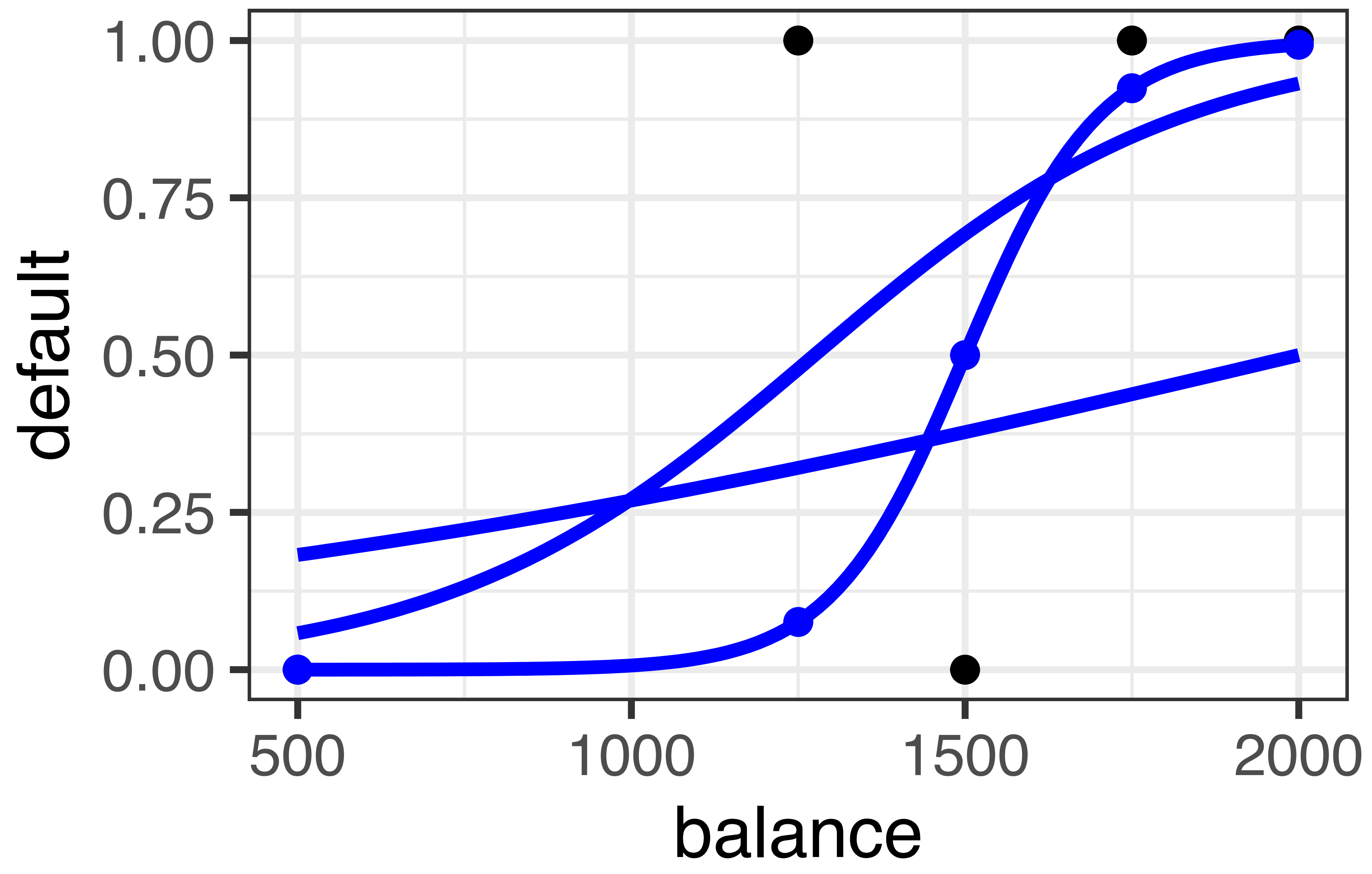

β_0	β_1	Predicted probabilities			$\mathscr{L}(\beta_0, \beta_1)$
-2.0	0.001	0.8	×	0.3×0.6×0.4×0.5	= 0.03
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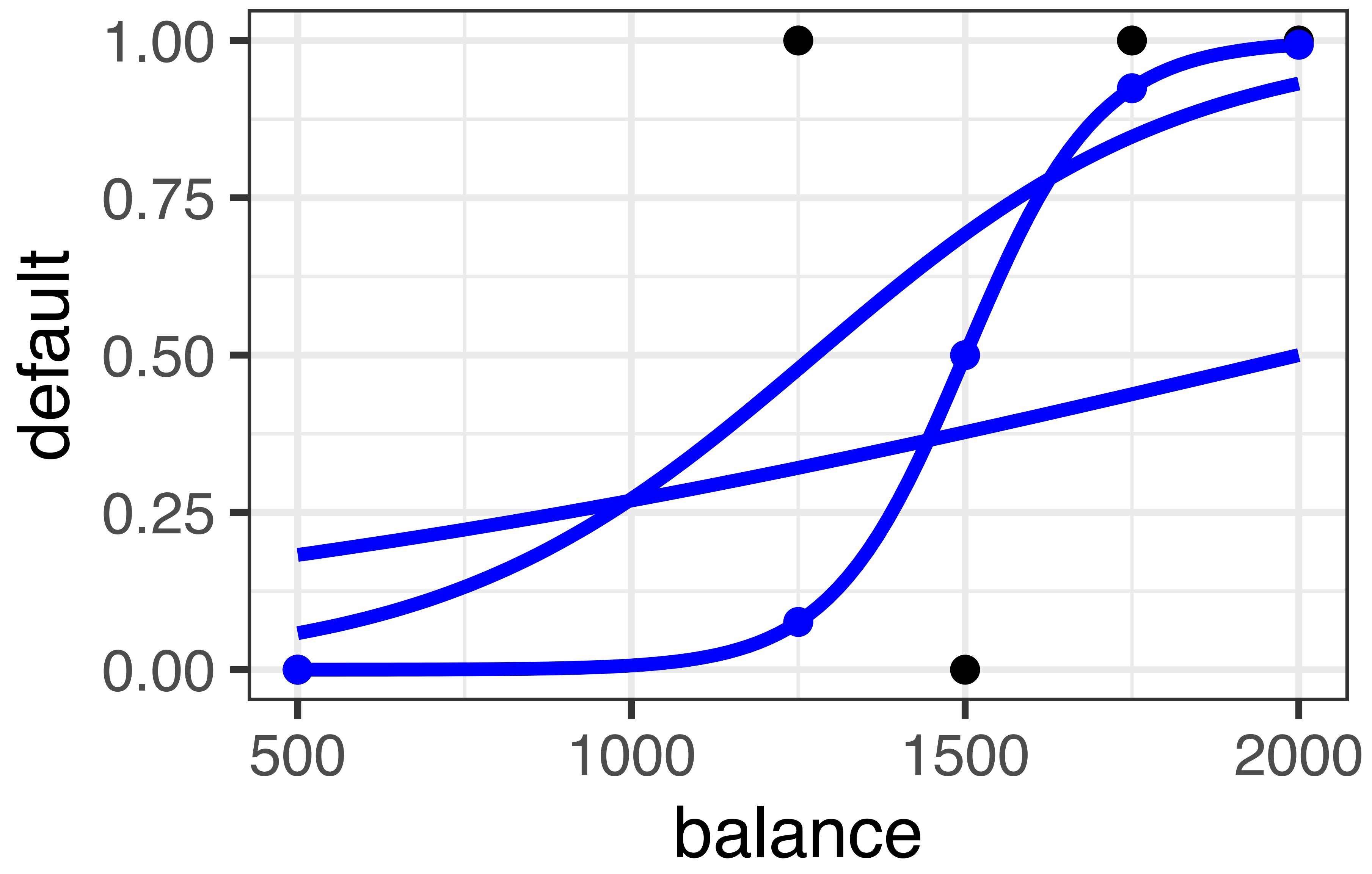

β_0	β_1	Predicted probabilities			$\mathscr{L}(\beta_0, \beta_1)$
-2.0	0.001	0.8	$\times$	$0.3\times0.6\times0.4\times0.5$	= 0.03
-4.6	0.004	0.9	$\times$	$0.5\times0.3\times0.8\times0.9$	= 0.1
-15.0	0.01	1.0		0.1 0.5 0.9 1.0	

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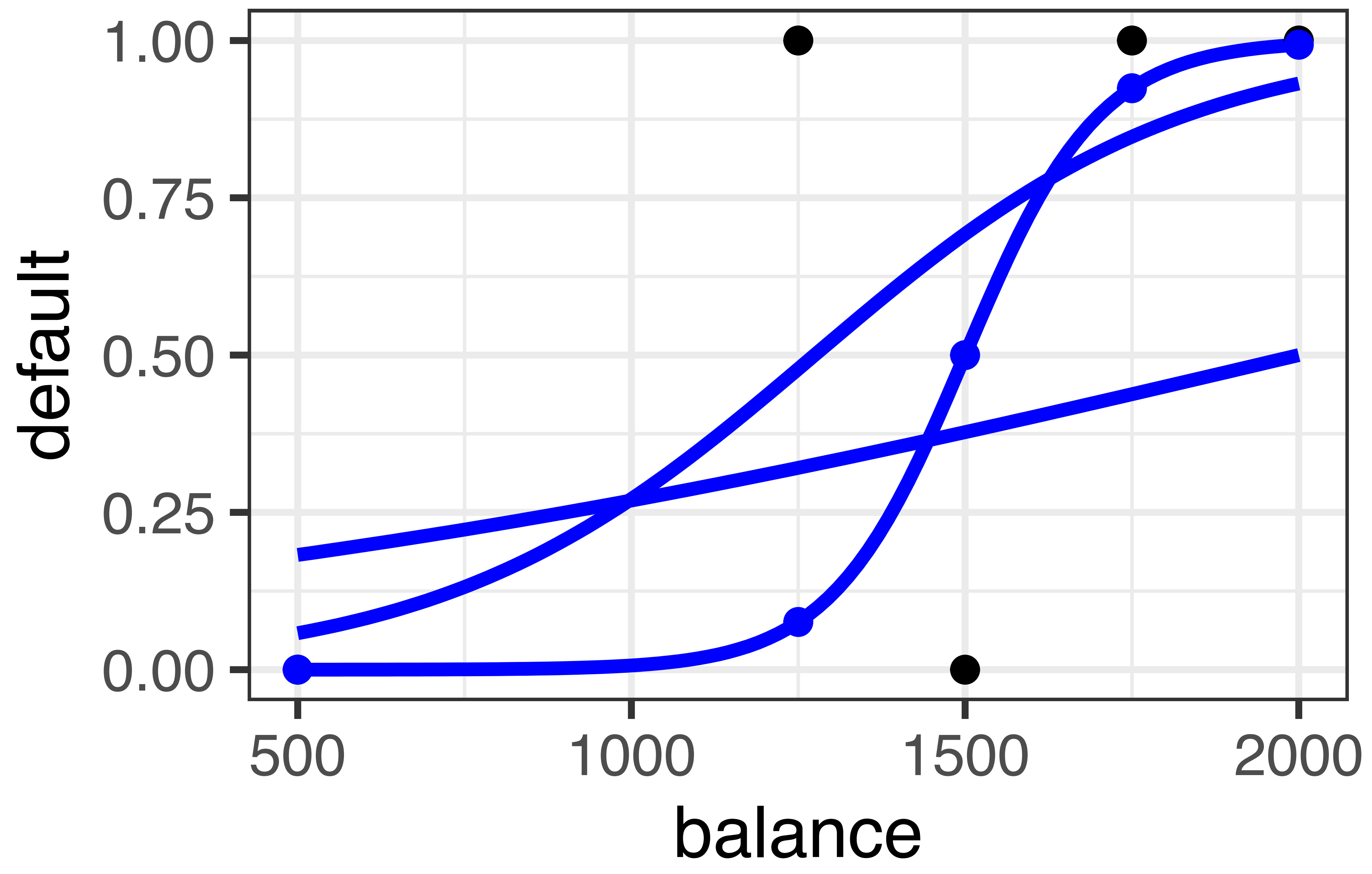

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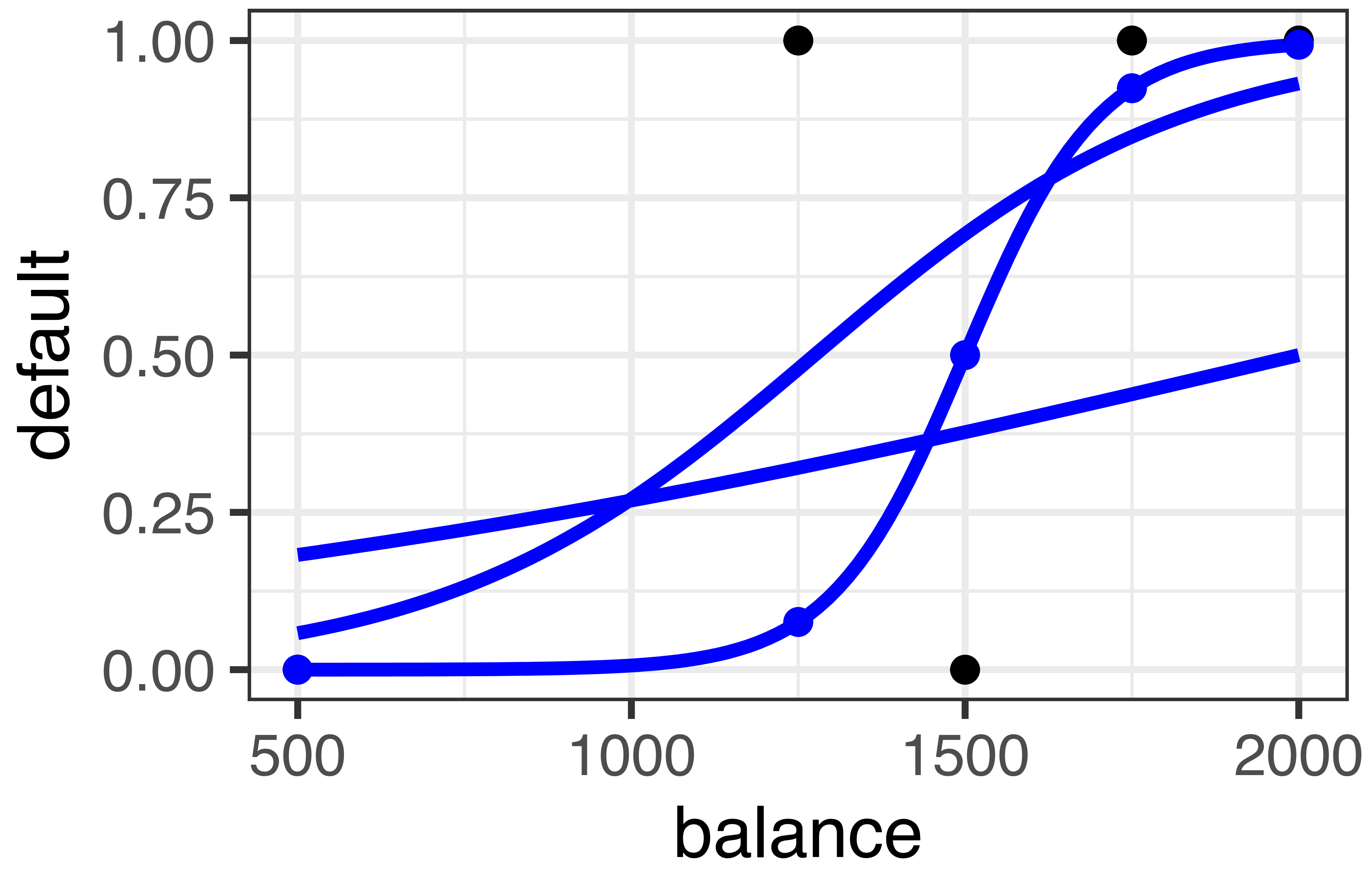

β_0	β_1	Predicted probabilities			$\mathscr{L}(\beta_0, \beta_1)$
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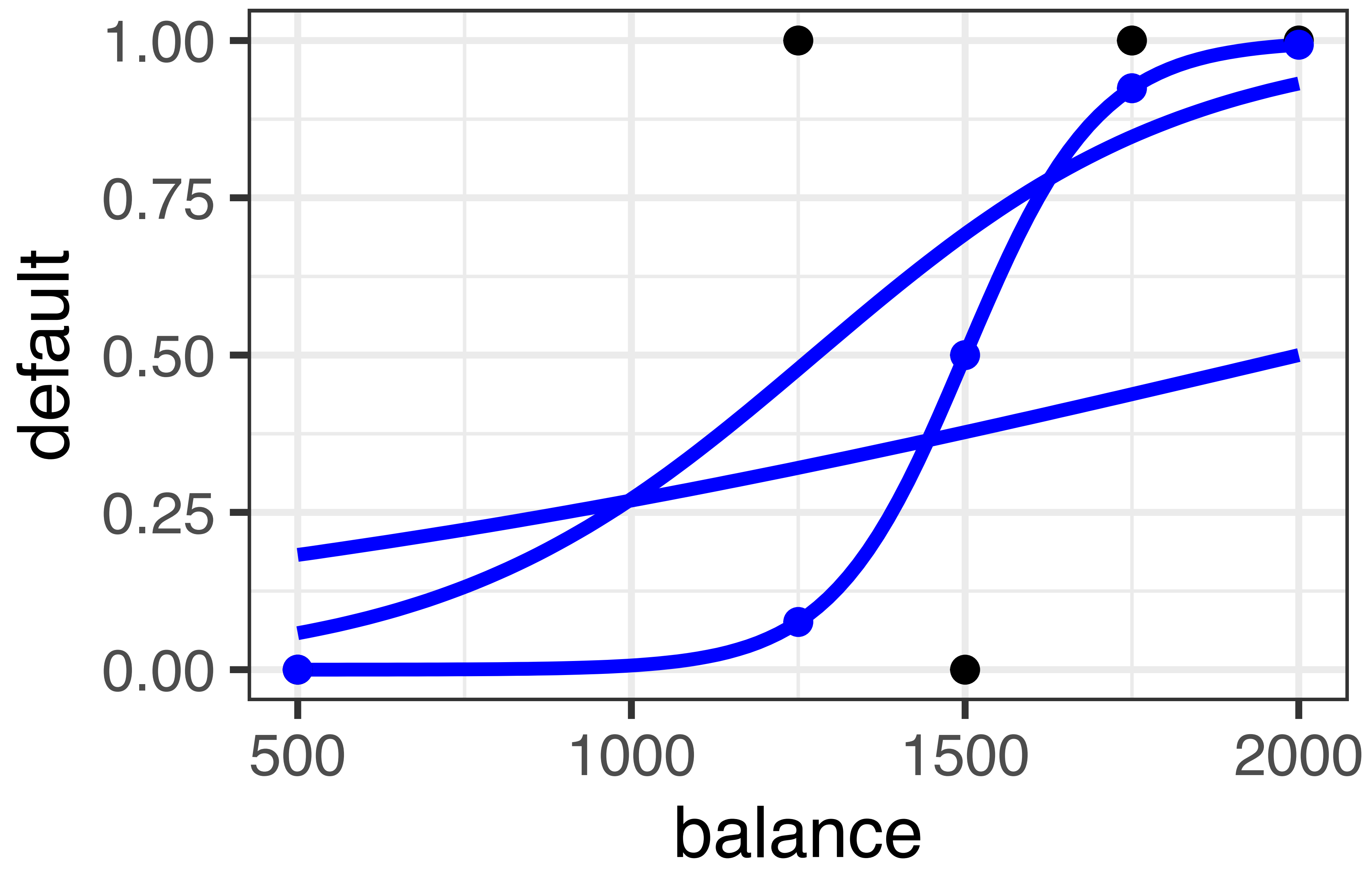

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-4.6	0.004	0.9	×	0.5×0.3×0.8×0.9	= 0.1
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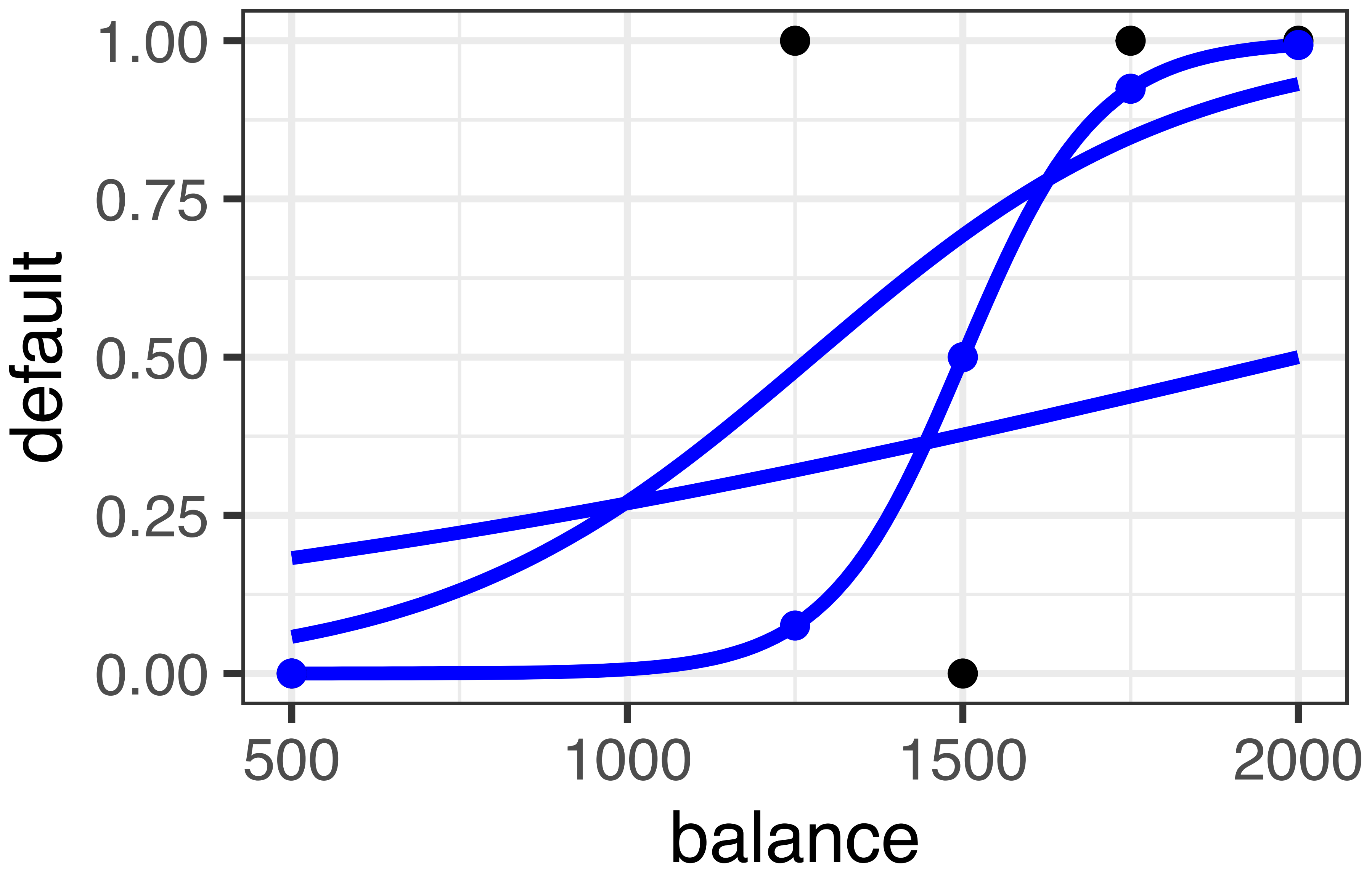
Mathematical expression for logistic likelihood

Maximum likelihood estimation

Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:

The maximum likelihood estimate (MLE) $(\widehat{\beta}_0, \widehat{\beta}_1)$ is defined as the maximizer of $\mathscr{L}(\beta_0, \beta_1)$.



β_0	β_1	Predicted probabilities			$\mathscr{L}(\beta_0, \beta_1)$
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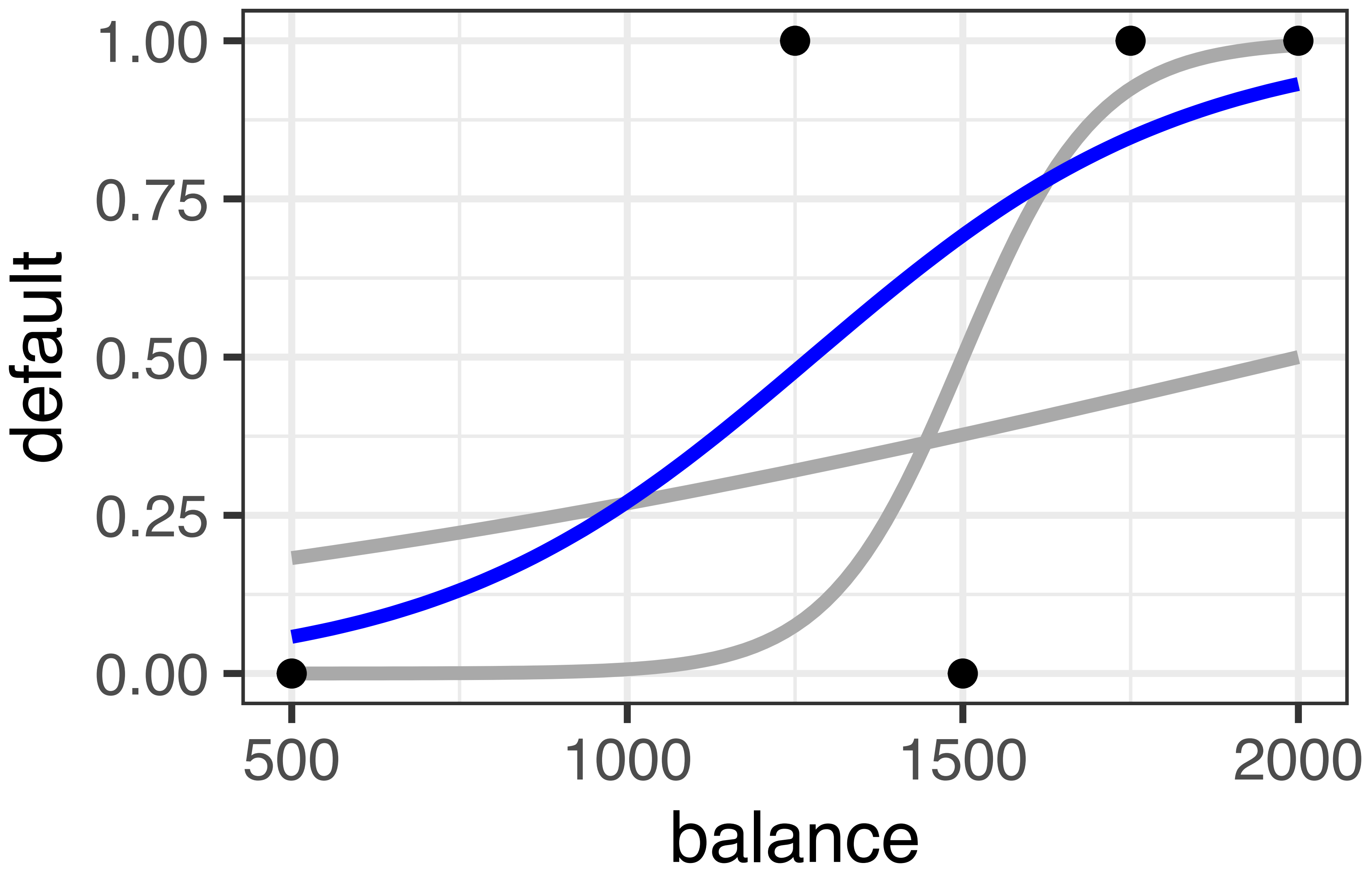

β_0	β_1	Predicted probabilities			$\mathscr{L}(\beta_0, \beta_1)$
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-4.6	0.004	0.9	×	0.5×0.3×0.8×0.9	= 0.1
-15.0	0.01	1.0	×	0.1×0.5×0.9×1.0	= 0.05

Mathematical expression for logistic likelihood

Maximum likelihood estimation

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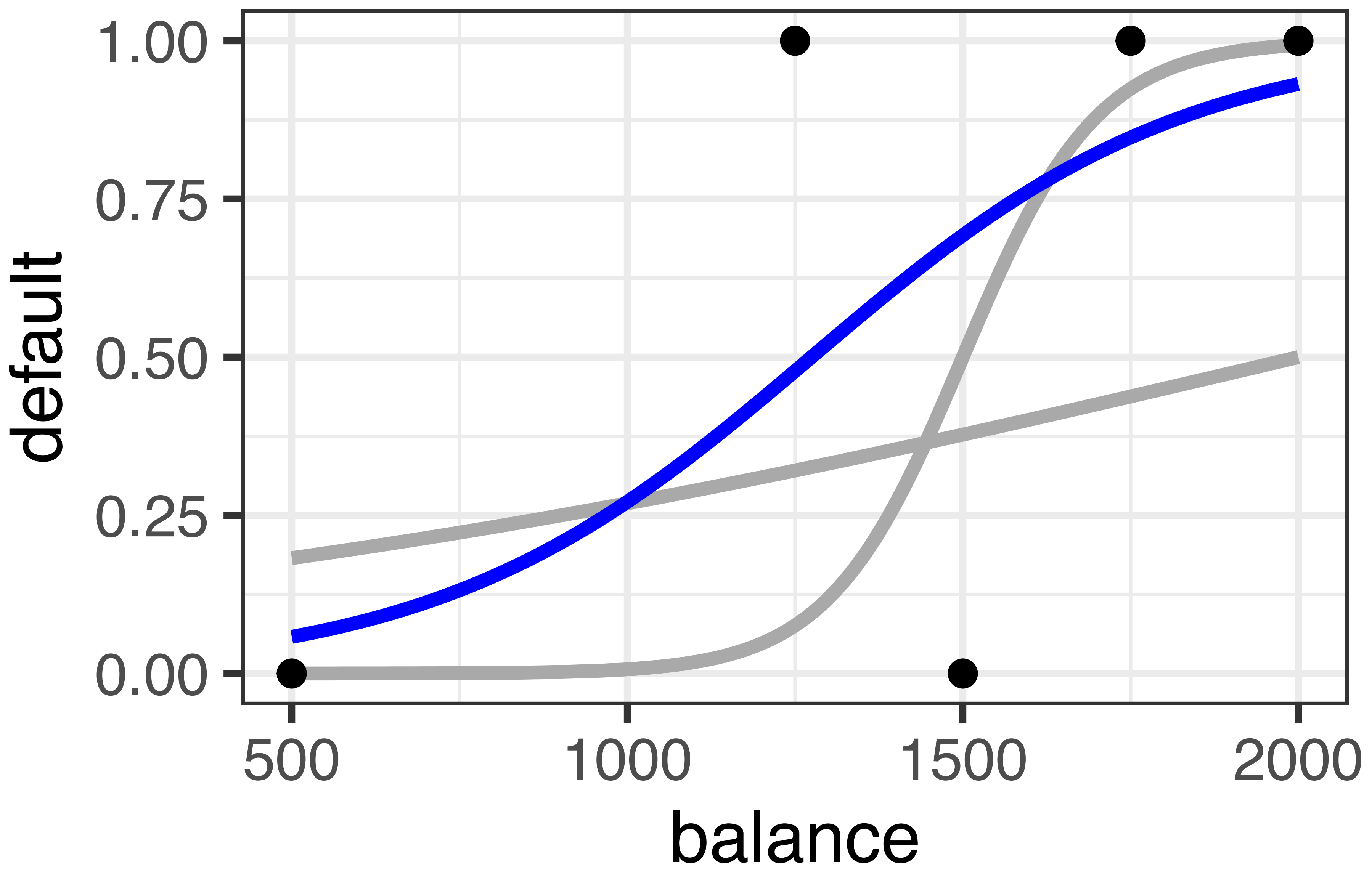


β_0	β_1	Predicted probabilities			$\mathscr{L}(\beta_0, \beta_1)$
-2.0	0.001	0.8	×	0.3×0.6×0.4×0.5	= 0.03
$(\hat{\beta}_0, \hat{\beta}_1) =$ (-4.6	0.004)	0.9	×	0.5×0.3×0.8×0.9	= 0.1
-15.0	0.01	1.0	×	0.1×0.5×0.9×1.0	= 0.05

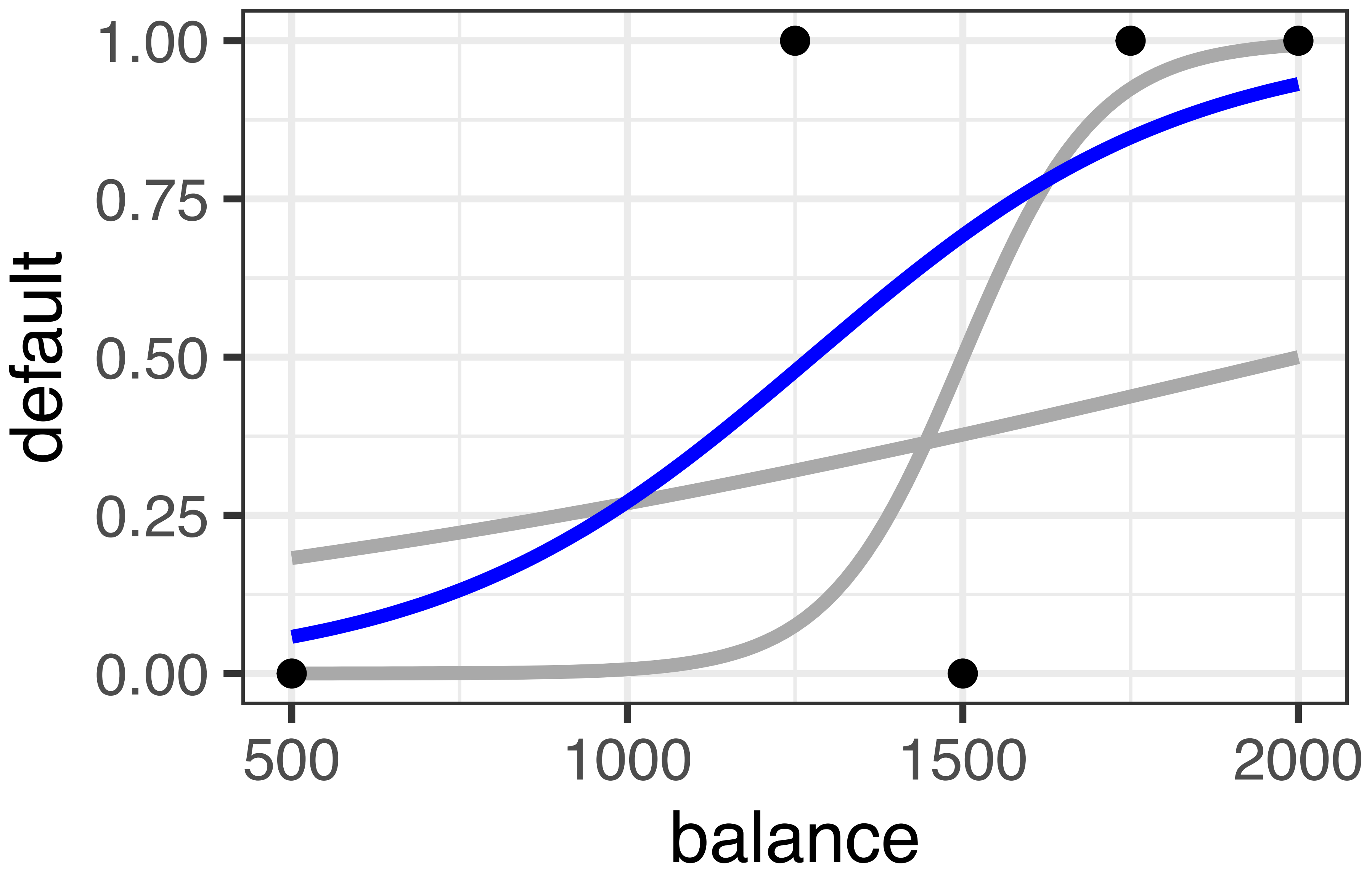
Mathematical expression for logistic likelihood

Maximum likelihood estimation

Given candidate parameters (β_0, β_1), we define the likelihood $\mathscr{L}(\beta_0, \beta_1)$ as the probability of observing the data under the corresponding model:

The maximum likelihood estimate (MLE) $(\widehat{\beta}_0, \widehat{\beta}_1)$ is defined as the maximizer of $\mathscr{L}(\beta_0, \beta_1)$.

It cannot be written in closed form; it is found via iterative algorithm.



	β_0	β_1	Predicted probabilities			$\mathscr{L}(\beta_0, \beta_1)$
	-2.0	0.001	0.8	×	0.3×0.6×0.4×0.5	= 0.03
$(\widehat{\beta}_0, \widehat{\beta}_1) =$	(-4.6 ,	0.004)	0.9	×	0.5×0.3×0.8×0.9	= 0.1
	-15.0	0.01	1.0	×	0.1×0.5×0.9×1.0	= 0.05

Mathematical expression for logistic likelihood

Multiple logistic regression

Like with linear regression, can include multiple features, e.g.

$$
\begin{aligned}
&\mathbb{P}[\text{default} \mid \text{student}, \text{balance}, \text{income}] \\
&\qquad = \text{logistic}(\beta_0 + \beta_1 \cdot \text{student} + \beta_2 \cdot \text{balance} + \beta_3 \cdot \text{income})
\end{aligned}
$$

The logistic regression likelihood, as well as the maximum likelihood estimates $(\widehat{\beta}_0, \widehat{\beta}_1, \widehat{\beta}_2, \widehat{\beta}_3)$ are defined analogously.

Interpreting logistic regression coefficients

$$\mathbb{P}[\text{default}] = \text{logistic}(\beta_0 + \beta_1 \cdot \text{student} + \beta_2 \cdot \text{balance} + \beta_3 \cdot \text{income})$$

Interpreting logistic regression coefficients

$$\mathbb{P}[\text{default}] = \text{logistic}(\beta_0 + \beta_1 \cdot \text{student} + \beta_2 \cdot \text{balance} + \beta_3 \cdot \text{income})$$

For given (student, balance, income),
suppose $\mathbb{P}[\text{default}] = 1/4$.

Interpreting logistic regression coefficients

$$\mathbb{P}[\text{default}] = \text{logistic}(\beta_0 + \beta_1 \cdot \text{student} + \beta_2 \cdot \text{balance} + \beta_3 \cdot \text{income})$$

$$\Downarrow$$

For given (student, balance, income), suppose $\mathbb{P}[\text{default}] = 1/4$.

$$\log \frac{\mathbb{P}[\text{default}]}{1 - \mathbb{P}[\text{default}]} = \beta_0 + \beta_1 \cdot \text{student} + \beta_2 \cdot \text{balance} + \beta_3 \cdot \text{income}$$

Interpreting logistic regression coefficients

$$\mathbb{P}[\text{default}] = \text{logistic}(\beta_0 + \beta_1 \cdot \text{student} + \beta_2 \cdot \text{balance} + \beta_3 \cdot \text{income})$$

$$\Downarrow$$

For given (student, balance, income), suppose $\mathbb{P}[\text{default}] = 1/4$.

$$\log \frac{\mathbb{P}[\text{default}]}{1 - \mathbb{P}[\text{default}]} = \beta_0 + \beta_1 \cdot \text{student} + \beta_2 \cdot \text{balance} + \beta_3 \cdot \text{income}$$

log-odds (the score from before)

Interpreting logistic regression coefficients

$$\mathbb{P}[\text{default}] = \text{logistic}(\beta_0 + \beta_1 \cdot \text{student} + \beta_2 \cdot \text{balance} + \beta_3 \cdot \text{income})$$

$\Downarrow$ For given (student, balance, income), suppose $\mathbb{P}[\text{default}] = 1/4$.

$$\log \frac{\mathbb{P}[\text{default}]}{1 - \mathbb{P}[\text{default}]} = \beta_0 + \beta_1 \cdot \text{student} + \beta_2 \cdot \text{balance} + \beta_3 \cdot \text{income}$$

log-odds (the score from before)

Interpreting logistic regression coefficients

$$\mathbb{P}[\text{default}] = \text{logistic}(\beta_0 + \beta_1 \cdot \text{student} + \beta_2 \cdot \text{balance} + \beta_3 \cdot \text{income})$$

$$\Downarrow$$

For given (student, balance, income), suppose $\mathbb{P}[\text{default}] = 1/4$.

$$\log \frac{\mathbb{P}[\text{default}]}{1 - \mathbb{P}[\text{default}]} = \beta_0 + \beta_1 \cdot \text{student} + \beta_2 \cdot \text{balance} + \beta_3 \cdot \text{income}$$

log-odds (the score from before)

Then, odds = 1:3 = 1/3 and log-odds = log(1/3) ≈ -1.

Interpreting logistic regression coefficients

$$\mathbb{P}[\text{default}] = \text{logistic}(\beta_0 + \beta_1 \cdot \text{student} + \beta_2 \cdot \text{balance} + \beta_3 \cdot \text{income})$$

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New odds are $e^1 \approx 2.7 = 2.7 : 1$, so new prob is 2.7/3.7 ≈ 0.7.
Odds went from e^{-1} (1/3) to e^1 (2.7), increase by factor of $e^2 \approx 7.5$.

Classification via logistic regression

$$\text{default} = \begin{cases} \text{Yes}, & \text{if } \widehat{\mathbb{P}}[\text{default}] \geq 0.5; \\ \text{No}, & \text{if } \widehat{\mathbb{P}}[\text{default}] < 0.5. \end{cases}$$

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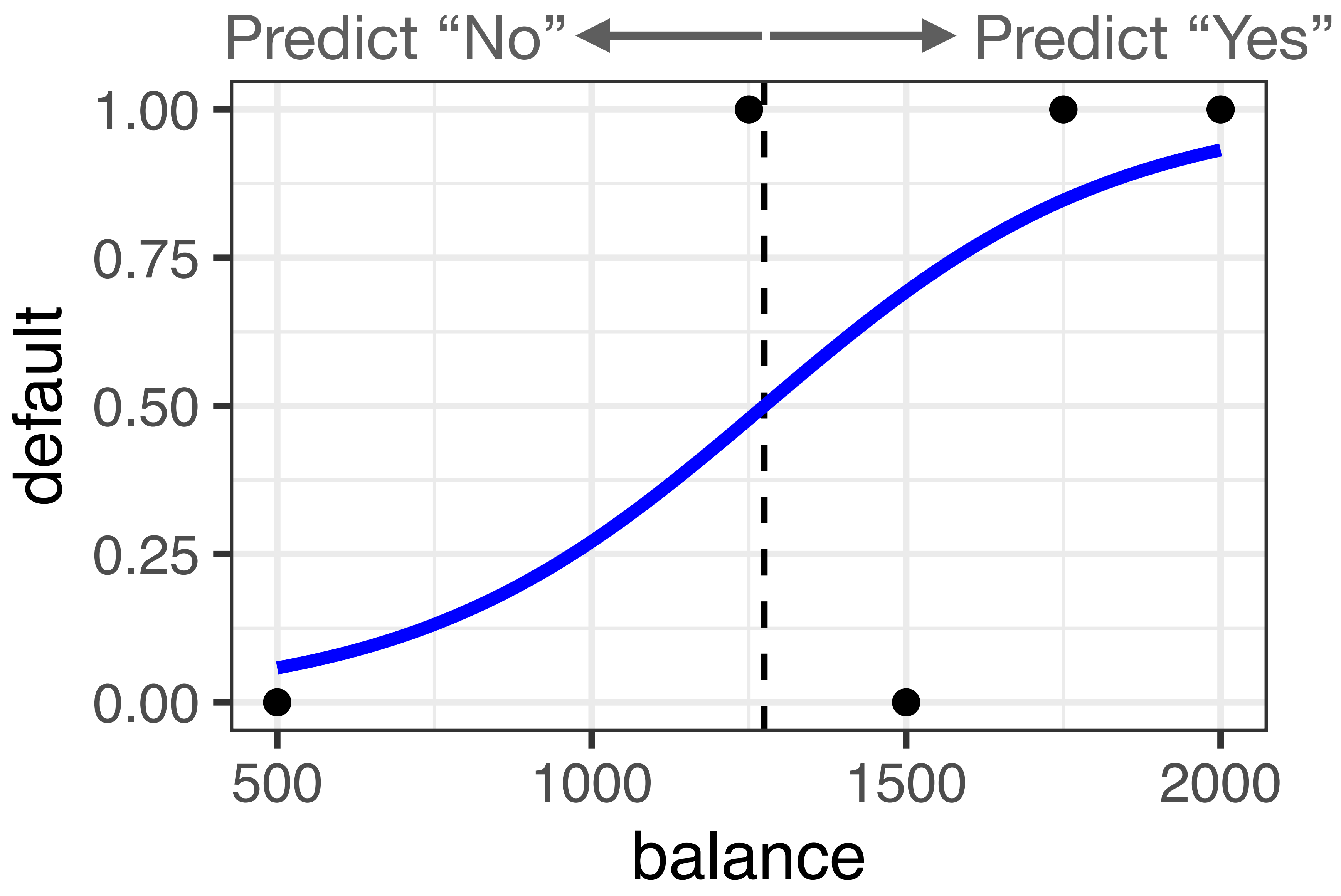

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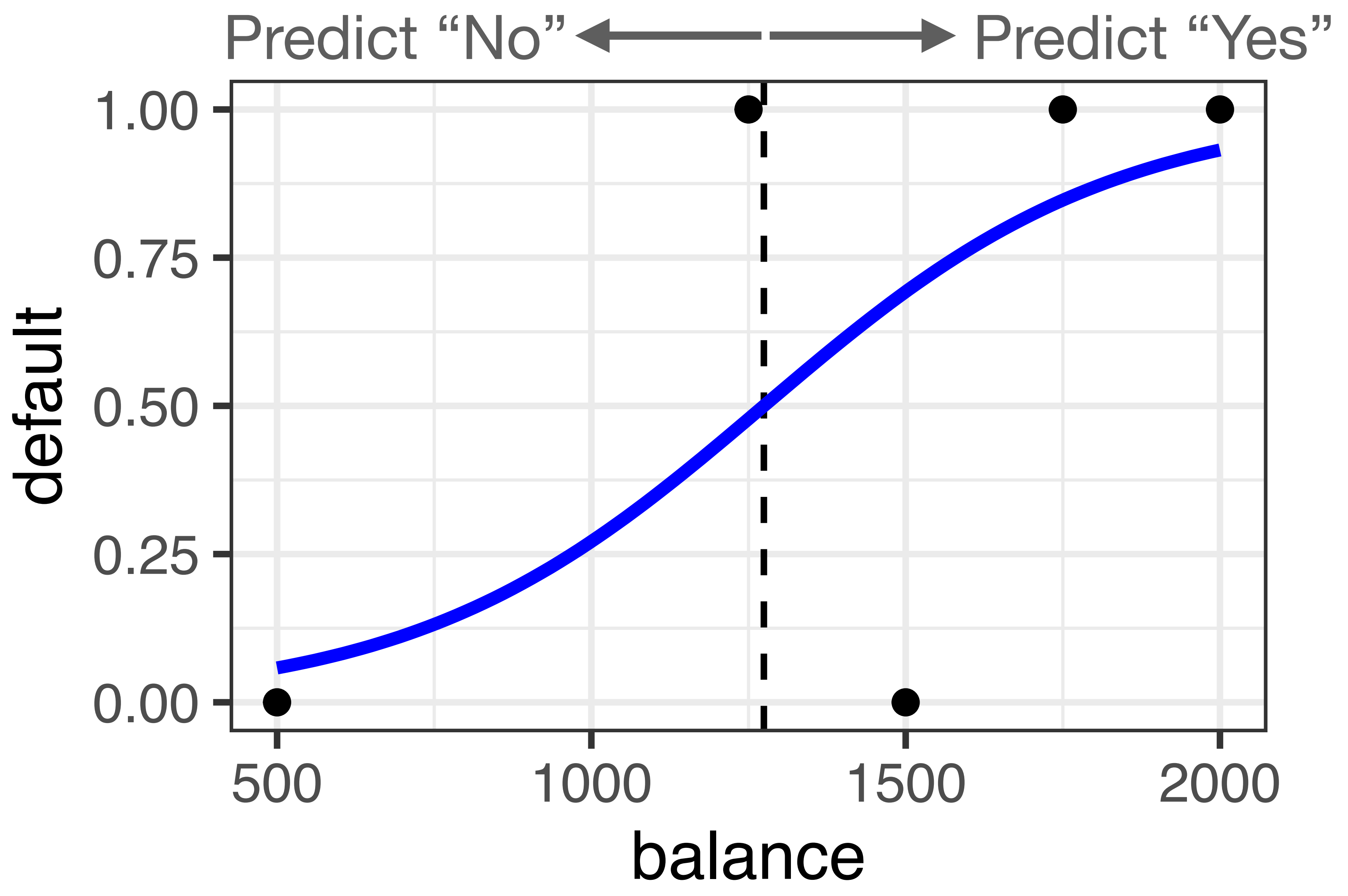

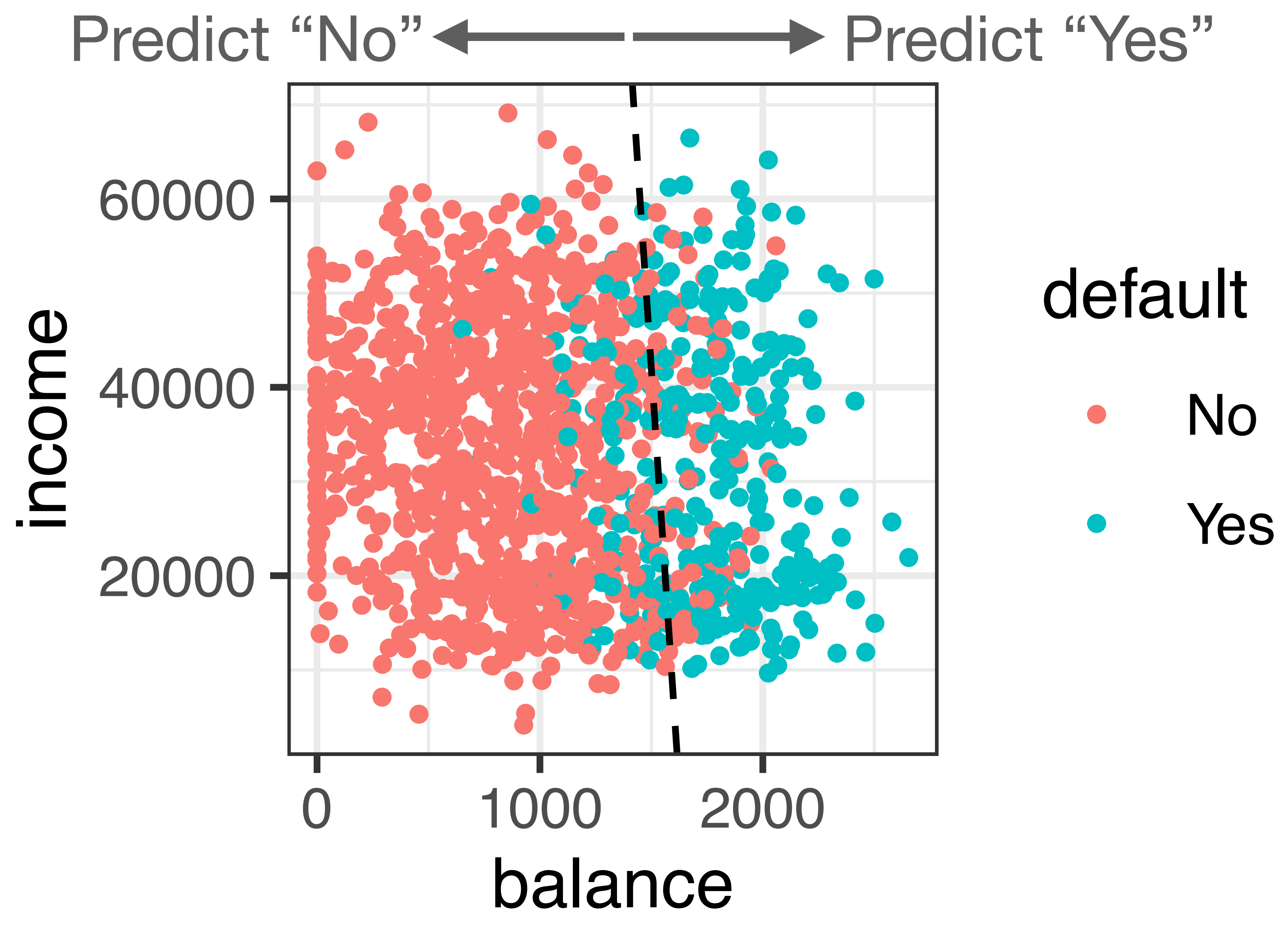

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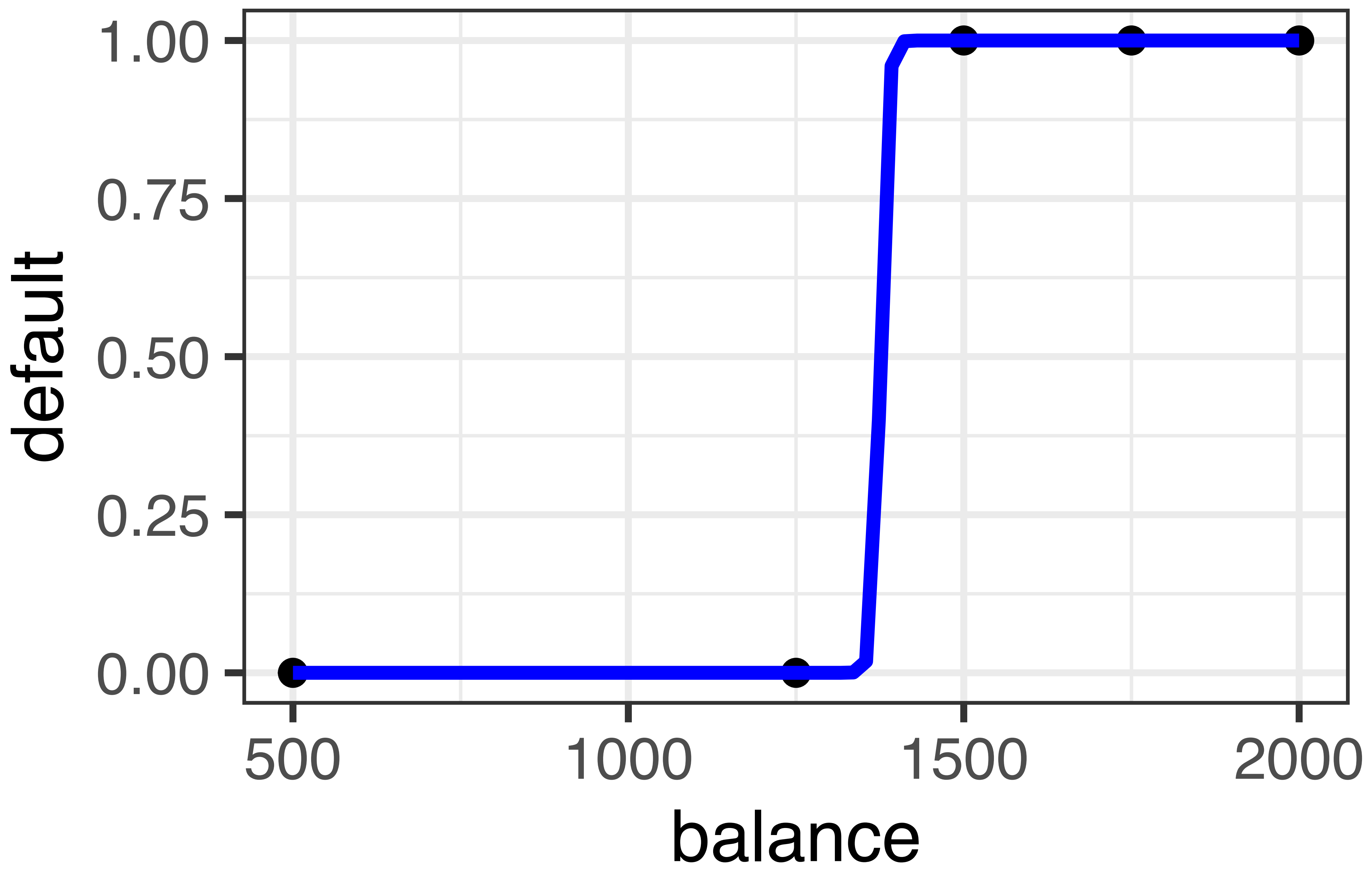
When the two classes of response variable can be perfectly separated in feature space, logistic regression solution undefined, though perfect predictions possible.



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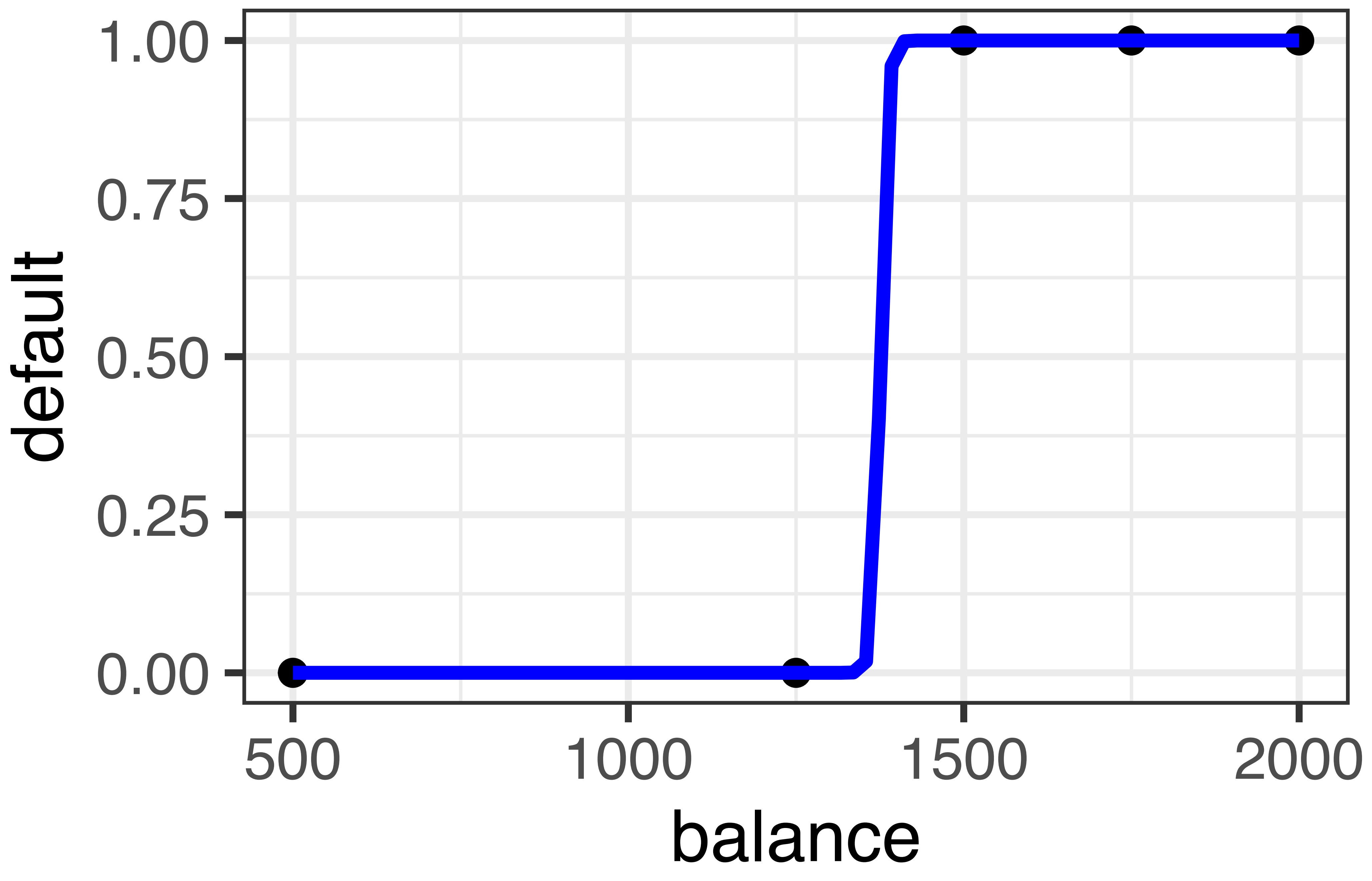

A similar phenomenon occurs in linear regression under perfect multicollinearity: The coefficient estimates are undefined but good prediction still possible.

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[Quiz Practice]

Mathematical expression for logistic likelihood

Data

default	balance	P[default = 1]	P[observed]
1	\$1250	$\frac{e^{\beta_0+\beta_1\cdot 1250}}{1+e^{\beta_0+\beta_1\cdot 1250}}$	$\frac{e^{\beta_0+\beta_1\cdot 1250}}{1+e^{\beta_0+\beta_1\cdot 1250}}$
0	\$500	$\frac{e^{\beta_0+\beta_1\cdot 500}}{1+e^{\beta_0+\beta_1\cdot 500}}$	$\frac{1}{1+e^{\beta_0+\beta_1\cdot 500}}$
1	\$2000	$\frac{e^{\beta_0+\beta_1\cdot 2000}}{1+e^{\beta_0+\beta_1\cdot 2000}}$	$\frac{e^{\beta_0+\beta_1\cdot 2000}}{1+e^{\beta_0+\beta_1\cdot 2000}}$
1	\$1750	$\frac{e^{\beta_0+\beta_1\cdot 1750}}{1+e^{\beta_0+\beta_1\cdot 1750}}$	$\frac{e^{\beta_0+\beta_1\cdot 1750}}{1+e^{\beta_0+\beta_1\cdot 1750}}$
0	\$1500	$\frac{e^{\beta_0+\beta_1\cdot 1500}}{1+e^{\beta_0+\beta_1\cdot 1500}}$	$\frac{1}{1+e^{\beta_0+\beta_1\cdot 1500}}$

$$\mathscr{L}(\beta_0, \beta_1) = \frac{e^{\beta_0+\beta_1\cdot 1250}}{1+e^{\beta_0+\beta_1\cdot 1250}} \times \frac{1}{1+e^{\beta_0+\beta_1\cdot 500}} \times \frac{e^{\beta_0+\beta_1\cdot 2000}}{1+e^{\beta_0+\beta_1\cdot 2000}} \times \frac{e^{\beta_0+\beta_1\cdot 1750}}{1+e^{\beta_0+\beta_1\cdot 1750}} \times \frac{1}{1+e^{\beta_0+\beta_1\cdot 1500}}$$